Effect of electrical conductivity and magnetization on the biomagnetic fluid flow over a stretching sheet

M.G.Murtaza, E.E. Tzirtzilakis and M. Ferdows

Abstract. The Biomagnetic Fluid Flow (BFD) (blood) over a stretching sheet in the presence of magnetic field is studied. For the mathematical formulation of the problem both magnetization and electrical conductivity of blood are taken into account and consequently both principles of Magnetohydrodynamics (MHD) and FerroHydroDynamics (FHD) are adopted. The physical problem is described by a coupled, nonlinear system of ordinary differential equations subject to appropriate boundary conditions. This solution is obtained numerically by applying an efficient numerical technique based on finite differences method. The obtained results are presented graphically for different values of the parameters entering into the problem under consideration. Emphasis is given to the study of the effect of the MHD and FHD interaction parameters on the flow field. It is apparent that both parameters effect significantly on various characteristics of the flow and consequently neither electrical conductivity nor magnetization of blood could be neglected.

Key words Stretching sheet, Biomagnetic fluid, Ferrohydrodynamics, Magnetohydrodynamics, ferrofluid, magnetic fluid, magnetization.

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1. Introduction

Biomagnetic fluid dynamics (BFD) is a relatively new area of fluid mechanics. Numerous applications have been proposed in bioengineering and medical science some of them include cancer tumor treatment by using magnetic hyperthermia or development of magnetic devices for cell separation [1, 2, 3].

BFD is the study of the effect of an applied magnetic field on biological fluid flow. An initial model of BFD was developed by Haik et al. and is actually based on the principles of Ferrohydrodynamics (FHD) [4]. According to this formulation, blood is considered as an electrically non conducting magnetic fluid and the flow is affected by the magnetization of the fluid in the magnetic field. Thus, the arising force is due to magnetization and depends on the existence of a spatially varying magnetic field. However, blood also possesses properties of an electrically conducting fluid due to the ions in the plasma. The flowing ions produce a slight electric current which interacts with magnetic fields. The formulation of electrically conducting fluids is made by adopting the principles of the well–known MagnetoHydroDynamics (MHD) which in contrast to FHD ignores the effect of polarization and magnetization [5]. In order to formulate the entire magnetic properties of blood i.e. electrical conductivity along with polarization an extended BFD model was developed. This model is consistent with the properties of MHD as well as with those of FHD and also includes the energy equation [6].

The shear-driven flow over a stretching sheet constitutes a classical physical problem first studied by Crane in 1970 for a Newtonian fluid [7]. Later, Anderson derived an exact similarity solution for velocity and pressure of the magnetohydrodynamic flow past a stretching sheet [8]. The study of MHD flow over a stretching sheet still constitutes a topic of current ongoing research. The radiation effects on the MHD flow near the stagnation point of a stretching sheet was studied by Jat and Chaudhary and Pop et al. [9, 10]. Das et al. studied the unsteady MHD flow of nanofluids over an accelerating convectively heated stretching sheet in the presence of a transverse magnetic field with heat source/sink [11]. The MHD flow of a viscous liquid film over a stretching sheet under different non–linear stretching velocities was studied by Dandapat et. al. [12]. Finally, a characteristic study concerning applications of MHD flow problems to hemodynamics is that of the steady incompressible viscoelastic and electrically conducting fluid flow and heat transfer in a parallel plate channel with stretching walls in the presence of a magnetic field [13].

Furthermore, analogous FHD flows over a stretching sheet have been investigated as well. A classical study Flow of a heated ferrofluid over a stretching sheet in the presence of a magnetic dipole is that of Anderson and Valnes [14]. Recently, Zeeshan et al. studied the effect of thermal radiation and heat transfer on the flow of ferromagnetic fluid on a stretching sheet. The appropriate combination of non–magnetic viscous base fluid, magnetic solid and surfactant composes magnetic fluid in the presence of magnetic dipole [15]. Finally, Tzirtzilakis and Kafoussias studied the three-dimensional laminar and steady boundary layer flow of an electrically non–conducting and incompressible magnetic fluid, with low Curie temperature and moderate saturation magnetization, over an elastic stretching sheet. It was also assumed that the magnetization of the fluid varied with the magnetic field strength H and the temperature T [16].

As far as the BFD flow over a stretching sheet is concerned the first work has been carried out by Tzirtzilakis and Kafoussias which was the study of a biomagnetic fluid flow over a stretching sheet with non–linear temperature dependent magnetization [17]. Moreover, Tzirtzilakis and Tanoudis have presented a numerical method for the study of laminar incompressible two dimensional biofluid over a stretching sheet with heat transfer. It was assumed that the magnetization of the fluid varied with the magnetic field strength H and the temperature T [18]. Recently, Misra and Shit studied the BFD flow of a non–Newtonian viscoelastic fluid over a stretching sheet under the influence of an applied magnetic field generated by a magnetic dipole. The magnetization of the fluid is considered to vary linearly with temperature as well as the magnetic field intensity [19].

To the authors' knowledge all the above mentioned BFD flows over a stretching sheet have been studied using either the formulation consistent with the principles of FHD or the formulation consistent with the principles of MHD. So, the present study concerns the flow of biomagnetic fluid over a stretching sheet, in the present of an applied magnetic field using the extended BFD model incorporating both FHD and MHD formulations [6]. The magnetization is considered to vary with the temperature and the magnetic field strength intensity and the biofluid is treated as an electrically conducting magnetic fluid which is also exhibits magnetization. The formulation of the problem is obtained by an analogous manner presented in previous studies [14, 17] and the numerical solution is obtained by applying an efficient numerical technique based on the common finite difference method [20]. The obtained results for critical flow characteristics like velocity, pressure and temperate as well as rate of heat transfer, skin friction or pressure on the stretching sheet are presented graphically for specific parameters entering into the problem under consideration. Special detailed analysis is performed for the variation of these physical quantities with the FHD and MHD interaction parameters which formulate the forces arising due to magnetization and the electrical conductivity, respectively.

2. Mathematical Formulation

Let us consider the viscous, steady, two–dimensional, laminar flow of an incompressible and electrically conducting biomagnetic fluid past a flat elastic sheet which is stretched with a velocity proportional to distance i.e. $u = cx$, where c is a dimensional constant. The temperature of the stretched sheet T_w is kept fixed and the temperature of the fluid far away from the sheet is T_c, where $T_c > T_w$. The fluid is confined to the half space above the sheet and magnetic dipole is located at distance d below the sheet, giving rise to a magnetic field of sufficient strength to saturate the biomagnetic fluid. The flow configuration is shown schematically at figure 1.

Figure 1: Flow configuration of the flow field

Under the above assumptions the equations governing the flow under consideration are [5, 21]:

Continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

Momentum equations:

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu_0 M\frac{\partial H}{\partial x} - \sigma B_y^2 u + \sigma B_x B_y v + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{2}$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu_0 M\frac{\partial H}{\partial y} - \sigma B_x^2 v + \sigma B_x B_y u + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right). \tag{3}$$

Energy equation:

$$\rho C_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) + \mu_0 T\frac{\partial M}{\partial T}\left(u\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y}\right) - \sigma B^2 u^2 = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \mu\left[2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2\right], \tag{4}$$

subject to the boundary conditions:

$$y = 0: \quad u = cx, \quad v = 0, \quad T = T_w \tag{5}$$

$$y \to \infty: \quad u = 0, \quad T = T_c, \quad p + 1/2\rho q^2 = const. \tag{6}$$

In the above equations $\mathbf{q} = (u, v)$ is the dimensional velocity, p is the pressure, ρ is the biomagnetic fluid density, σ is the electrical conductivity, μ is the dynamic viscosity, C_p the specific heat at constant pressure, k the thermal conductivity, μ_0 is the magnetic permeability, $\mathbf{H} = (H_x, H_y)$ is the magnetic field strength and B is the magnetic induction ($\mathbf{B} = \mu_0 \mathbf{H} \Rightarrow (B_x, B_y) = \mu_0(H_x, H_y)$).

The terms $-\sigma B_y^2 u + \sigma B_x B_y v$ and $-\sigma B_x^2 v + \sigma B_x B_y u$ in (2) and (3), respectively, represent the Lorentz force per unit volume towards the x and y directions respectively, whereas the term $-\sigma B^2 u^2$ in the energy equation (4) represents the Joule heating. The joule heating term in the energy equation is simplified and assumed to be generated mainly by the x–component of the velocity which is dominant in the boundary layer flow. These terms arise due to the electrical conductivity of the fluid and are known in MHD [5, 11, 12, 13]. The terms $\mu_0 M \frac{\partial H}{\partial x}$ and $\mu_0 M \frac{\partial H}{\partial y}$ in (2) and (3), respectively, represent the components of the magnetic force per unit volume and depend on the existence of the magnetic gradient on the corresponding x and y directions. The second term on the left–hand side of the energy equation (4), accounts for heating due to the adiabatic magnetization. These terms are known from FHD [6, 5, 16, 17, 18].

The magnetic dipole gives rise to a magnetic field, sufficiently strong to saturate the biofluid, and its scalar potential is given by Andersson and Valnes [14]

$$V(x, y) = \frac{\alpha}{2\pi} \frac{x}{x^2 + (y + d)^2} \tag{7}$$

Thus the magnitude $||H|| = H$ of the magnetic field intensity by

$$H(x, y) = \left(H_x^2 + H_y^2\right)^{\frac{1}{2}} = \frac{\gamma}{2\pi} \frac{x}{x^2 + (y + d)^2} \tag{8}$$

where $\gamma = \alpha$ and H_x, H_y are the component of the magnetic field $\vec{H} = (H_x, H_y)$ given by

$$H_x\left(x,y\right) = -\frac{dV}{dx} = \frac{\gamma}{2\pi}\frac{x^2-\left(y+d\right)^2}{\left[x^2+\left(y+d\right)^2\right]^2}, \tag{9}$$

$$H_y\left(x,y\right) = -\frac{dV}{dy} = \frac{\gamma}{2\pi}\frac{2x\left(y+d\right)}{\left[x^2+\left(y+d\right)^2\right]^2}. \tag{10}$$

Following analogous manipulations to previous studies [14, 16, 17, 18] the gradients of the magnetic field strength, can be obtained from equation (8) after having expanded in powers of x and retained terms up to x^2, thus

$$\begin{cases} \dfrac{\partial H}{\partial x} \approx -\dfrac{\gamma}{2\pi}\dfrac{2x}{\left(y+d\right)^4}, \\ \dfrac{\partial H}{\partial y} \approx \dfrac{\gamma}{2\pi}\left[-\dfrac{2}{\left(y+d\right)^3}+\dfrac{4x^2}{\left(y+d\right)^5}\right]. \end{cases} \tag{11}$$

The magnetic field intensity H, can be expressed by an analogous manner, as

$$H\left(x,y\right) \approx \frac{\gamma}{2\pi}\left[\frac{1}{\left(y+d\right)^2}-\frac{x^2}{\left(y+d\right)^4}\right] \tag{12}$$

The above relations of the magnetic field strength H and its gradients i.e. (12) and (11), respectively, are valid close to region where $x=0$ and are used for the further transformation of the system of the governing equations.

Moreover, under the assumption that the applied magnetic field $\vec{H}$ is sufficiently strong to saturate the biomagnetic fluid, the magnetization M is generally determined by the fluid temperature and magnetic field intensity H. There is a variety of equations that can be used for the variation of the magnetization under the equilibrium assumption [6]. In this study the relation of Matsuki et al derived experimentally is adopted. This relation expresses the magnetization as a function of the magnetic field strength intensity H and the temperature of the fluid T [22].

$$M = KH(T_c - T) \tag{13}$$

where K is a constant called pyromagnetic coefficient and T_c is the Curie temperature. The above relation for the magnetization M has also proposed for the formulation of BFD [6] and used for stretching sheet flow problems [16, 18].

3. Transformation of Equations

Following Anderson and Valnes [14] we introduce the following non–dimensional coordinates

$$\begin{cases} \xi(x) = \left(c\dfrac{\rho}{\mu}\right)^{\frac{1}{2}} x, \\ \eta(y) = \left(c\dfrac{\rho}{\mu}\right)^{\frac{1}{2}} y \end{cases} \tag{14}$$

and the variables

$$\Psi(\xi,\eta) = \left(\frac{\mu}{\rho}\right) \xi f(\eta), \tag{15}$$

$$p(\xi,\eta) = \frac{P}{c\mu} = -P_1 - \xi^2 P_2(\eta) \tag{16}$$

$$\Theta(\xi,\eta) = \frac{\mathrm{T_c} - \mathrm{T}}{\mathrm{T_c} - \mathrm{T_w}} = \Theta_1 + \xi^2 \Theta_2(\eta) \tag{17}$$

where $\Psi(\xi,\eta)$, $\Theta(\xi,\eta)$ and $p(\xi,\eta)$ are the stream function and the dimensionless temperature and pressure, respectively.

The velocity components can be calculate as

$$\begin{cases} u = \dfrac{\partial \Psi}{\partial y} = cxf'(\eta), \\ v = -\dfrac{\partial \Psi}{\partial y} = -\dfrac{\mu}{\rho} f \left(c\dfrac{\rho}{\mu}\right)^{\frac{1}{2}}. \end{cases} \tag{18}$$

Substituting equations (11) to (17) into the momentum equations in (2) and (3) and the energy equation (4) and equating the coefficients of equal power of ξ up to ξ^2 then we get the following system of differential equations

$$f''' + ff'' - (f')^2 + 2P_2 - \frac{2\alpha^2\beta\Theta_1}{(\eta+\alpha)^6} - Mf' = 0, \tag{19}$$

$$P_1' - f'' - ff' - \frac{2\alpha^2\beta\Theta_1}{(\eta+\alpha)^5} - Mf = 0, \tag{20}$$

$$P_2' + \frac{6\alpha^2\beta\Theta_1}{(\eta+\alpha)^7} - \frac{2\alpha^2\beta\Theta_2}{(\eta+\alpha)^5} = 0, \tag{21}$$

$$\Theta_1'' + Prf\Theta_1' + \frac{2\beta\lambda\alpha^2(\Theta_1 - T_\varepsilon)f}{(\eta+\alpha)^5} + 2\Theta_2 - 4\lambda(f')^2 = 0, \tag{22}$$

$$\Theta_2'' - \lambda(f'')^2 - \Pr(2f'\Theta_2 - f\Theta_2') - 2\beta\lambda\alpha^2(\Theta_1 - T_\varepsilon)\left(\frac{f'}{(\eta+\alpha)^6} + \frac{3f}{(\eta+\alpha)^7}\right) + \\ + \frac{2\beta\lambda\alpha^2(\Theta_2 - T_\varepsilon)f}{(\eta+\alpha)^5} + M\lambda(f')^2 = 0. \tag{23}$$

And the boundary conditions (5) and (6) are transformed to:

$$\eta = 0: \quad f' = 1, \quad f = 0, \quad \Theta_1 = 1, \quad \Theta_2 = 0, \tag{24}$$

$$\eta \to \infty: \quad f' \to 0, \quad \Theta_1 \to 0, \quad \Theta_2 \to 0, \quad P_1 \to -P_\infty, \quad P_2 \to 0. \tag{25}$$

The dimensional parameters appearing in the above governing equations are:

$Pr = \frac{\mu c_p}{k}$	Prandtl number
$T_\varepsilon = \frac{T_c}{T_c - T_w}$	Dimensionless temperature parameter
$\lambda = \frac{c\mu^2}{\rho k(T_c - T_w)}$	Viscous dissipation parameter
$\beta = \frac{\gamma}{2\pi}\frac{\mu_0 KH(0,0)(T_c - T_w)\rho}{\mu^2}$	Ferromagnetic interaction parameter
$M = \frac{\sigma\mu_0^2 H^2}{c\rho}$	Magnetohydrodynamic interaction parameter
$\alpha = \left(\frac{c\rho}{\mu}\right)^{\frac{1}{2}} d$	Dimensionless distance

The ferromagnetic interaction parameter arises in the governing equations due to the magnetization (polarization) of the fluid and is consistent to the FHD properties. If one set $M \neq 0$ and $\beta = 0$ to the governing equations (19)-(23) then the polarization is “switched off” and the governing equations along with the corresponding boundary conditions correspond to the MHD flow over a stretching sheet. On the other hand the Magnetohydrodynamic interaction parameter arises in the governing equations due to the electrical conductivity of the biofluids. If now one set $M = 0$ and $\beta \neq 0$ then the effect of the electrical conductivity is omitted and the governing equations along with the corresponding boundary conditions formulate the pure FHD flow over a stretching sheet. It is clear that if $M = \beta = 0$ then the set of equations corresponds to a pure hydrodynamic flow.

The system of equations (19)-(23) subject to the boundary conditions (24) and (25), constitute a six parameter $(\alpha, \beta, \lambda, M, Pr, T_\varepsilon)$ coupled and non–linear system of ordinary differential equations, describing the BFD flow over a stretching sheet when the fluids exhibits both electrical conductivity and magnetization which is given as a function of temperature T and the magnetic field strength H.

4. Numerical Method

For the numerical solution of the problem under consideration we apply an approximate technique that has better stability characteristics than classical Runge–Kutta combined with a shooting method, is simple, accurate and efficient. The essential features of this technique are the following: (i) It is based on the common finite difference method with central differencing (ii) on a tridiagonal matrix manipulation and (iii) on an iterative procedure [20]. For reasons of completeness of this study we demonstrate the application of this methodology for the numerical solution of the system of equations (19), (22) and (23), subject to the boundary conditions (24) and (25).

The momentum Equation (19) can be written as

$$f''' + ff'' - f'^2 - \mathrm{M}f' = \frac{2\alpha^2\beta\Theta_1}{(n+\alpha)^6} - 2P_2 \tag{26}$$

The above equations can be considered as a second order linear differential equation by setting $y(x) = f'(\eta)$ provided that P_2 and $f(\eta)$ are considered known functions. In this case equation (26) can be written as

$$(f')'' + f(f')' - (f' + M)\, f' = \frac{2\alpha^2\beta\Theta_1}{(n+\alpha)^6} - 2P_2$$

which is of the form

$$P(x)\, y''(x) + Q(x)\, y'(x) + R(x)\, y(x) = S(x) \tag{27}$$

where $P(x) = 1$, $Q(x) = f(n)$, $R(x) = -f'(n) - M$, $S(x) = \frac{2\alpha^2\beta\Theta_1}{(n+\alpha)^6} - 2P_2$.

In an analogous manner all equations of the system can be reduced in this form of equation (27) except for equation (20) and (21) which are already first order differential equations. Equation (26) can be solved by a common finite difference method, based on central differencing and tridiagonal matrix manipulation.

To start the solution procedure, we assume initial guesses (distribution curves) for $f'(\eta)$ and $P_2(\eta)$ between $\eta = 0$ and $\eta = \eta_\infty$ ($\eta \to \infty$) which satisfy the boundary conditions (24) and (25). For this problem indicative initial guesses are

$f'(\eta) = \left(1 - \frac{\eta}{\eta_\infty}\right)$, $\Theta_1 = \left(1 - \frac{\eta}{\eta_\infty}\right)$ and $\Theta_2 = 0.5\left(\frac{\eta}{\eta_\infty}\right)\left(1 - \frac{\eta}{\eta_\infty}\right)$

The $f(\eta)$ distribution is obtained by the integration from $f'(\eta)$ curve. The next step is to consider the f, P_2 and Θ_1 known and to determine a new estimation for $f'(\eta)$, ($f'_n ew$) by solving the non-linear equation (27) using the above method. The distribution is updated by the integration of new f' curve. These new profiles of f' and f are then used for new inputs and so on. In this way the momentum equation (26) and consequently (19) is solved iteratively until convergence up to a small quantity ε is attained.

After $f(\eta)$ is obtained the solution of the energy equation (22) with boundary condition (24) and (25) is solved by using the same algorithm, but without iteration now as for as equation (22) is linear. Equation (22) is

$$\Theta_1'' + Prf\Theta_1' + \frac{2\beta\lambda\alpha^2(\Theta_1 - T_\varepsilon)f}{(\eta+\alpha)^5} + 2\Theta_2 - 4\lambda(f')^2 = 0$$

which can be written as

$$\Theta_1'' + Prf\Theta_1' + \frac{2\beta\lambda\alpha^2 f}{(\eta+\alpha)^5}\Theta_1 = \frac{2\beta\lambda\alpha^2 f}{(\eta+\alpha)^5}T_\varepsilon - 2\Theta_2 + 4\lambda(f')^2. \qquad (28)$$

Equation (28) by setting $y(\eta) = \Theta_1(\eta)$ is again a second order linear differential equation of the form

$$P(x)\,y''(x) + Q(x)\,y'(x) + R(x)\,y(x) = S(x) \qquad (29)$$

where $P(x) = 1$, $Q(x) = Prf(n)$, $R(x) = \frac{2\beta\lambda\alpha^2 f}{(\eta+\alpha)^5}$, $S(x) = \frac{2\beta\lambda\alpha^2 T_\varepsilon f}{(\eta+\alpha)^5} - 2\Theta_2 + 4\lambda(f')^2$.

Considering f, f' and θ_2 known, we obtain a new approximation $\Theta_1 new$ for Θ_1 and this process continues until convergence up to a small quantity ε is attained and finally we obtain Θ_1.

Hereafter the energy equation (23) with boundary condition (24) and (25) is solved. Equation (23) can be written as

$$\Theta_2'' + \Pr f\Theta_2' + \left(\frac{2\beta\lambda\alpha^2 f}{(\eta+\alpha)^5} - 2\Pr f'\right)\Theta_2 =$$
$$2\beta\lambda\alpha^2\left(\Theta_1 - T_\varepsilon\right)\left(\frac{f'}{(\eta+\alpha)^6} + \frac{3f}{(\eta+\alpha)^7}\right) + \frac{2\beta\lambda\alpha^2 f T_\varepsilon}{(\eta+\alpha)^5} - M\lambda\left(f'\right)^2 + \lambda\left(f''\right)^2 \quad (30)$$

Equation (30) is a second order linear differential equation by setting $y(\eta) = \Theta_2(\eta)$ which is of the form (27) with

$P(x) = 1$, $Q(x) = Prf(n)$, $R(x) = \frac{2\beta\lambda\alpha^2 f}{(\eta+\alpha)^5} - 2Prf'(n)$, and

$S(x) = 2\beta\lambda\alpha^2\left(\Theta_1 - T_\varepsilon\right)\left(\frac{f'}{(\eta+\alpha)^6} + \frac{3f}{(\eta+\alpha)^7}\right) + \frac{2\beta\lambda\alpha^2 f T_\varepsilon}{(\eta+\alpha)^5} - M\lambda(f')^2 + \lambda(f'')^2$

Considering f, f', f'' and Θ_1 known we calculate the new approximation $\Theta_2(new)$ for Θ_2 and continue this iteration until convergence up to a small quantity is attained and finally we obtain Θ_2. Considering now Θ_1 and Θ_2 known, we obtained a new estimate for P_1 and P_2 (equations (20)–(21)). Next the computational procedure reverts to its starting point i.e. the aforementioned solution of equation (26) using the most recent calculations of the distributions $f'(\eta)$, $P(\eta)$ and $\Theta_1(\eta)$ as inputs. This process is continuing until final convergence of the solution is attained.

In order to apply to our numerical computation a proper step size $h = \Delta\eta = 0.01$ and appropriate η_∞ value as $y \to \eta$ must be determined. By "trial and error" we set $\eta_\infty = 6$ and the tolerance between the iterations is set at $\varepsilon = 10^{-4}$ defined as $\varepsilon = \max_{i=1,N}\left(\left|\frac{f_{old}(i) - f_{new}(i)}{f_{old}(i)}\right|\right)$. Computations were also performed for $\Delta\eta = 0.001$ and no significant differences were found.

5. Results and discussion

For the derivation of the numerical solution it is necessary to assign some numerical values to the parameters involved in the problem under consideration. In this study we adopt case scenarios also discussed in previous studies [23, 24] according to which the fluid is blood with density $\rho = 1050kgm^{-3}$ and viscosity $\mu = 3.2 \times 10^{-3}kgm^{-3}s^{-3}$. The electrical conductivity of blood is $\sigma = 0.8sm^{-1}$. The temperature of the plate is $T_w = 37^oC$ whereas the temperature of the fluid is $T_c = 41^oC$. For these values the temperature number T_ε is equal to 78.5 and the viscous dissipation number is $\lambda = 6.4 \times 10^{-14}$. Although the viscosity μ the specific heat under constant pressure C_p and thermal conductivity k of any fluid and hence of the fluid is blood, are temperature dependent the prandtl number can be considered constant. Thus, for the temperature range considered in this problem $C_p = 3.9 \times 10^3 jKg^{-1}K^{-1}$ and $k = 0.5jm^{-1}s^{-1}K^{-1}$ and hence $Pr = 25$.

As far as the values of the magnetic parameters M and β are concerned there exist extended discussions in various studies [6, 23, 24]. Especially the biomagnetic interaction parameter can take a quite large range of values depending by the magnetic field gradient. For β in the present study we adopt values in the range of $0 - 10$ used also in previous studies [17, 18, 19]. For the magnetic parameter M the range that could be adopted is also large and could reach the value of 600 for very strong magnetic fields [13]. In this study we perform calculations for the range $0-10$ for M. The above ranges of the magnetic parameters albeit correspond to low values of the magnetic field strength we will see that result to considerable changes in the flow field comparable to the hydrodynamic case which is given for $M = \beta = 0$.

In order to compare the obtained numerical results with others documented in literature, computations were carried out by setting $M = 0$ and for $\beta = 0, 2$ and 5. The results are identical with those obtained by Tzirtzilakis and Tanoudis [18] as well as with those obtained by Tzirtzilakis and Kafoussias [17] for the values of the critical exponent $\delta = 0$ and $\beta = 0, 5$ and the corresponding values of the parameters refereed in that study. It is noted that the results obtained by Tzirtzilakis and Tanoudis [18] have been also validated with results obtained by Anderson and Valnes [14] and are in accordance for the hydrodynamic case ($M = \beta = 0$) with the results obtained by Crane [7]. Furthermore, additional comparisons were performed for the MHD case with analytical results provided by Anderson for the dimensionless stream function f, for $\beta = 0$ and $M = 5$ [8]. It is found that the absolute difference at all the points of calculation between the theoretical and numerical estimated value is less than 5×10^{-5}. It is noted that the Lorentz force at the study of Anderson is risen only due to the u-velocity component whereas, in the present study both velocity components are taken into account (see eqs (2) and (3)). This accordance between the present results and the analytical ones presented by Anderson indicate an interesting matter as far as the physical problem is concerned. The consideration of both velocity components in the Lorentz force does not significantly alter the flow field and equations (2) and (3) can be simplified. This is justified due to the fact that the u velocity component is dominant to the boundary layer flow field and the v–velocity component is insignificant to cause further changes in the Lorentz force. Moreover, the above results indicate that the simplification made in the energy equation concerning the joule heating term, i.e the consideration that only the u–velocity component gives rise to the joule heating, is valid.

Figure 2: Variation of the dimensionless velocity component $f'(\eta)$

From relation (18) it is apparent that $f'(\eta) = u/cx$. The function $f'(\eta)$ is called dimensionless velocity component and its variation is pictured at fig. 2. The curves are plotted for $M = \beta = 0$ which corresponds to pure hydrodynamic flow, $M = 0$, $\beta = 10$ which corresponds to pure FHD flow, $M = 5$, $\beta = 0$ which corresponds to pure MHD flow and finally for $M = 5$, $\beta = 10$ which correspond to the mixed FHD and MHD flow of the extended BFD model. It is observed that the dimensionless velocity is reduced considerably with the increment of β or M. Increment of β causes reduction of the dimensionless velocity. However, the major reduction of the velocity is observed with the increment of M and the differences by increasing β is negligible comparable to those occur by the increment of M.

Figure 3: Variation of the dimensionless temperature $\Theta_1(\eta)$

Fig. 3 shows the variation of the dimensionless temperature $\Theta_1(\eta)$ for the same values of M and β as with the dimensionless velocity above. Generally, the temperature in the flow field increases with the increment of the magnetic parameters M or β. Again the greater increment of Θ_1 occurs with the increment of M and when β increases smaller increments are noticed. The higher temperature distribution in the flow field is observed for the extended BFD case ($M = 5, \beta = 10$). The curve for $M = 0$, $\beta = 5$ is similar to the corresponding one obtained in previous studies [17, 18]. The calculations show that the dimensionless temperature $\Theta(\xi, \eta)$ is represented only by the function $\Theta_1(\eta)$ whereas $\Theta_2(\eta)$ is negligible. Namely, the calculated absolute values of the distribution of Θ_2 were less than 10^{-5} which is well below the accuracy of the numerical method used and thus, Θ_2 is practically zero.

Figure 4: Variation of the dimensionless relative pressure $\Delta P_1(\eta)$

The dimensionless pressure P_1 is estimated from integration of equation (20) under the boundary condition (25) i.e. $P_1 \to -P_\infty$. It is noted that this equation is not coupled to the rest of the system of the governing equations and is solved once at the end of the procedure. The boundary condition is derived from the initial set of the equations and the Bernoulli equation at conditions (6) holding far away from the stretching sheet. If one consider the relative pressure $\Delta P_1 = P_1 - P_\infty$ then equation (20) can be integrated for unknown function the relative pressure ΔP_1 under the boundary condition $\Delta P_1 \to 0$ as $\eta \to \infty$. The variation of the relative pressure ΔP_1 for various numbers M and β is shown at fig. 4. It is obtained that the determining factor of the reduction of the relative pressure is the parameter M. The arrows point the direction of increment for β at the intersection of the arrow and the graphs. When the parameter β increases, for a specific value of M, it is observed that the relative pressure also increases almost all over the flow field except a region close to the stretching sheet ($0 \leq \eta \lesssim 0.2$) where the opposite happens. It is noted that the decrement of the relative dimensionless P_1 is almost one order of magnitude for $M = 5$ close to the area of magnetic field.

Figure 5: Variation of the dimensionless pressure $P_2(\eta)$

Fig. 5 shows the variation of the dimensionless pressure P_2 with the magnetic parameters M and β. A general observation is that the variations of P_2 are limited close to the stretching sheet and for $0 \leq \eta \lesssim 0.5$. It is obtained that this time the important parameter for the increment of P_2 is β. For a specific value of β increment of M results to further small increment of P_2. The curve for $M = 0$, $\beta = 5$ is similar to the corresponding one obtained in the aforementioned previous studies [17, 18].

Figure 6: Variation of the dimensionless wall shear parameter $-f''(0)$: (A) with β, (B) with M.

Figure 7: Variation of the dimensionless relative wall pressure $\Delta P_1(0)$: (A) with β, (B) with M.

Another important parameters investigated in stretching sheet problems are the dimensionless wall shear parameter $f''(0)$ and the dimensionless wall heat transfer parameter $\Theta^*(0)$. These parameters are related to the local skin friction coefficient and the local rate of heat transfer respectively [18]. The variation of $-f''(0)$ is shown at figs 6(a) and 6(b). The increment of β leads to linear increment of $-f''(0)$. The line for $M = 0$ is similar to corresponding one obtained in previous studies [17, 18]. Moreover, further increment is observed if M increases for a specific value of β. The increment of $-f''(0)$ with M is not linear and is depicted at fig. 6(b). Increment of either M or β result to almost equivalent significant increment of $-f''(0)$.

Figures 7(a) and 7(b) show the variation of the dimensionless relative wall pressure $\Delta P_1(0)$ with β and various values of M and with M for $\beta = 0$, 5 and 10, respectively. The relative wall pressure $\Delta P_1(0)$ reduces linearly with the increment of β. It is apparent from Fig. 7(b) that the reduction of $\Delta P_1(0)$ is much greater with the increment of M than that caused by the increment of β. On the other hand from Fig. 8 it is obtained that the dimensionless wall pressure $P_2(0)$ increases linearly with the increment of β and the increment of M does not have significant effects in the flow field. The line for $M = 0$ is similar to the corresponding one obtained in the aforementioned previous studies [17, 18].

Figure 8: Variation of the dimensionless wall pressure $P_2(0)$ with β.

Figure 9: Variation of the wall heat transfer parameter $\Theta^*(0)$: (A) with β (B) with M

Another interesting parameter for the study of the thermal problem is the so called coefficient of the heat transfer rate at the wall (sheet) which is independent of the distance ξ and is defined by the ratio $\Theta^*(0) = \frac{\Theta'_1(0)}{\Theta'_1(0)|_{\mathrm{M}=\beta=0}}$. The variation of the wall heat transfer parameter $\Theta^*(0)$ with β and M is shown at Figs. 9a and 9b. For the case of the variation with β pictured at Fig. 9a, $\Theta^*(0)$ reduces linearly. The reduction is greater comparable to the hydrodynamic case ($M = \beta = 0$) as M increases. Figure 9b shows the variation of $\Theta^*(0)$ with M which for this case is not linear. It is generally obtained that the increment of M or β results to similar amount of reduction for this parameter. The maximum rate of heat transfer at the wall is attained for $\beta = 10$ and $M = 5$. The line for $M = 0$ at fig 9a is similar to the corresponding one obtained in previous study [17, 18].

6. Conclusions

For the problem of the BFD flow over a stretching sheet it is concluded that the electrical conductivity and the polarization of the fluid are both determining factors of the flow field. The dimensionless velocity of the fluid over the stretching sheet is reduced by the application of the magnetic field. This reduction is caused almost exclusively from the electrical conductivity whereas the reduction caused by the polarization is negligible. Analogous behavior is observed for the dimensionless temperature Θ_1. The effect of the electrical conductivity of the fluid prevails over the one caused by the polarization effect on the values of the dimensionless relative pressure ΔP_1 whereas the opposite is true for the dimensionless pressure P_2. As far as the very important characteristics of the flow on the stretching sheet are concerned the dimensionless wall shear parameter $-f''(0)$ is almost equally affected by the variation of M or β. Increment of β results to increment of $-f''(0)$. On the other hand the electrical conductivity plays the dominant role in the variation of dimensionless relative wall pressure $\Delta P_1(0)$ which reduces as M increases. Moreover, the dimensionless wall pressure P_2 is not affected by the increment of M and increases linearly with the increase of β. The coefficient of the heat transfer rate at the wall (sheet) $\Theta^*(0)$ decreases with the increase of β and/or M. The polarization has less, nonetheless significant effect on the variation of $\Theta^*(0)$ than the electrical conductivity of the biofluid. Overall, the adoption of the extended BFD model combining the principles of MHD and FHD is necessary to be adopted for the study of stretching sheet problems since for the values of the parameters used both electrical conductivity and polarization play important role in the formation of the flow field.

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M.G.Murtaza

M. Ferdows

Research group of Fluid Flow Modeling and Simulation,

Department of Applied Mathematics,University of Dhaka,

Dhaka-1000, Bangladesh.

E.E. Tzirtzilakis

Fluid Dynamics & Turbomachinery Laboratory,

Department of Mechanical Engineering

Technological Educational Institute of Western Greece

1 M. Aleksandrou Str, Koukouli,

26334 Patras, Greece

M.G.Murtaza, E.E. Tzirtzilakis and M. Ferdows