Numerical study of forced and free convective boundary layer flow of a magnetic fluid over a flat plate under the action of a localized magnetic field

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Abstract

The two – dimensional, steady, laminar, forced and free convective boundary layer flow of a magnetic fluid over a semi-infinite vertical plate, under the action of a localized magnetic field, is numerically studied. The magnetic fluid is considered to be water-based with temperature dependent viscosity and thermal conductivity. The study of the boundary layer is separated into two cases. In case I the boundary layer is studied near the leading edge, where it is dominated by the large viscous forces, whereas in case II the boundary layer is studied far from the leading edge of the plate where the effects of buoyancy forces increase. The numerical solution, for these two different cases, is obtained by an efficient numerical technique based on the common finite difference method. Numerical calculations are carried out for the value of Prandl number Pr = 49.832 (water-based magnetic fluid) and for different values of the dimensionless parameters entering into the problem and especially for the magnetic parameter Mn, the viscosity/temperature parameter Θ_r and the thermal/conductivity parameter S^*. The analysis of the obtained results show that the flow field is influenced by the application of the magnetic field as well as by the variation of the viscosity and the thermal conductivity of the fluid with temperature. It is hoped that they could be interesting for engineering applications.

Keywords: FHD, magnetic fluids, free-forced convective flow, FDM.

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1. Introduction

Ferro hydrodynamics (FHD) deals with the mechanics of magnetic fluid motion influenced by strong forces of magnetic polarization. A magnetic Ferro fluid consists of a stable colloidal dispersion of sub-domain magnetic particles in a liquid carrier. Magnetic fluids have been used commercially for a number of years in numerous devices, such as rotating shaft seals and exclusion seals, loud speakers, dampers, inclinometers etc, whereas magnetorheological fluids have achieved commercial use in hydraulical devices, loudspeakers and in grinding applications. According to Berkovski and Bashtovoy [1], numerous patents and a great number of scientific papers have been published, related to the preparation, properties and application of magnetic fluids [2] ~ [5]. In order to examine the flow of a magnetic fluid, under the action of an applied magnetic field, mathematical models have been developed by many investigators [6] ~ [9].

The two classical problems in fluid mechanics, namely the Blasius boundary layer flow along a flat plate and the stagnation point flow, were extended for a saturated Ferro fluid, under the combined influence of thermal and magnetic field gradients in [10]. The flow of a viscous Newtonian fluid past a linearly stretching surface in otherwise quiescent surroundings was first considered in [11]. Similar problems for a micro polar fluid or for inelastic power law fluids were studied by many authors [12] ~ [16]. The problem studied in [16] was extended in [17] by assuming that the magneto-thermo-mechanical coupling is not described by a linear function of temperature difference as in [16], but by a non linear one, the expression of which was used in [18]. Another classical problem in fluid mechanics is the free or the forced convective boundary layer flow, of a viscous incompressible fluid, over a vertical hot plate surface. In FHD, thermomagnetic convection, in conjunction with gravity-induced convection, is of fundamental importance for engineering applications. This type of problem was studied by many authors during the last seven years [19] ~ [23].

The aim of the present work is the numerical study of the two-dimensional, steady and laminar free-forced convective boundary layer flow of a magnetic fluid over a semi infinite vertical hot plate under the action of a localized magnetic field. To the authors' knowledge, this physical problem has not yet been studied. The magnetic field $\vec{H}$ is considered to be of sufficient strength to saturate the magnetic fluid and the magnetization of the fluid is considered to be a linear function of the magnetic field intensity. For the mathematical formulation of the problem, which is presented in Section 2, the variation of the viscosity μ of the fluid, as well as of its thermal conductivity k, with temperature T, is taken into account. For the whole study, two separate cases are considered. The first (**Case I**), concerns the study of the boundary layer near the leading edge, where the boundary layer is dominated by the large viscous forces. The second one (**Case II**), concerns the study of the boundary layer far from the leading edge of the plate, where the effects of buoyancy forces increase and play a significant role to the evolution of the boundary layer. For each case, the mathematical analysis of the problem is simplified by introducing suitable dimensionless coordinates and variables of transformation. The numerical solution of the coupled non linear system of partial differential equations (PDEs), with its boundary conditions, describing the problem under consideration for **Case I** and **Case II**, is obtained by an efficient numerical solution technique presented in Section 3. Finally, numerical results for the fundamental quantities of the flow field, such as the velocity and temperature profiles and the skin friction and heat transfer coefficient, are presented and analyzed in Section 4. The analysis of the obtained results indicates that application of a magnetic field in such a flow of a magnetic fluid could be interesting and useful for engineering applications.

2. Mathematical formulation of the problem

The forced and free convective, steady, two-dimensional, laminar boundary layer flow of a viscous incompressible and homogeneous magnetic fluid, over a semi-infinite vertical flat plate is considered.

In a Cartesian coordinate system $Oxyz$, the plate is located at $y=0,\ \ 0 \le \mathrm{x} < \infty,\ \ -\infty < \mathrm{z} < \infty$ with the positive x-axis in the upward direction. The magnetic fluid is considered to be electrically non-conducting and flowing parallel to the plate in the positive x-direction, with a constant free stream velocity u_∞. The temperature T_w of the plate is considered to be uniform, constant and greater than the free stream fluid temperature T_∞, far away from the hot plate. The fluid flow is subject to the action of a localized magnetic field $\vec{H}$ of sufficient strength to saturate the fluid. The magnetic field is generated by an electric current, with intensity I, going through an infinite thin wire placed parallel to the plate (to the z - axis), at a distance a from the origin of the Cartesian coordinate system $Oxyz$ and at a distance b below the plate. Therefore, the position of this wire is

$$(x_0, y_0, z) = (a, b, z) \quad a > 0\,,\ b < 0\,, \quad \mathrm{z} \in \mathrm{R}\,.$$

In such a case the magnitude $\left\|\vec{H}\right\|$ of the magnetic field intensity, is given by the expression

$$\left\|\vec{H}\right\| \ = \ H(x,y) = \frac{I}{2\pi}\frac{1}{[(x-a)^2 + (y-b)^2]^{1/2}}\,. \qquad (1)$$

For the formulation of the physical problem, the following assumptions are also made. Fluid property variations with temperature are limited to density, viscosity and thermal conductivity, with the density variation taken into account only as far as it effects the buoyancy term in the momentum equation (Boussinesq approximation). Under all the above assumptions, the two-dimensional laminar boundary layer forced and free convective flow of the magnetic fluid, past the semi-infinite vertical plate, in the Oxy plane, is governed by the following equations

Continuity

$$\frac{\partial u}{\partial x}+\frac{\partial \upsilon}{\partial y}=0, \qquad (2)$$

Momentum

$$u\frac{\partial u}{\partial x}+\upsilon\frac{\partial u}{\partial y}=\frac{1}{\rho_\infty}\frac{\partial}{\partial y}(\mu\frac{\partial u}{\partial y})+g\beta(T-T_\infty)+\frac{\mu_o}{\rho_\infty}M\frac{\partial H}{\partial x}, \qquad (3)$$

Energy

$$u\frac{\partial T}{\partial x}+\upsilon\frac{\partial T}{\partial y}-\frac{\mu_o}{\rho_\infty c_p}T\frac{\partial M}{\partial T}(u\frac{\partial H}{\partial x}+\upsilon\frac{\partial H}{\partial y})=\frac{1}{\rho_\infty c_p}\frac{\partial}{\partial y}(k\frac{\partial T}{\partial y})\,. \qquad (4)$$

In the above equations u,υ are the fluid velocity components in the x and y direction, respectively, g is the gravitational acceleration, β is the coefficient of thermal expansion of the fluid, T is the fluid temperature inside the boundary layer, ρ_∞ is the fluid density away from the hot plate, μ is the viscosity of the fluid, μ_0 is the magnetic permeability of vacuum, k is the thermal conductivity and c_p is the specific heat of fluid at constant pressure.

The boundary conditions of the problem are

$$y=0: \quad u=0, \quad \upsilon=0, \quad T=T_w, \qquad (5)$$

$$y\to\infty: \quad u=u_\infty, \quad T=T_\infty\,. \qquad (6)$$

The term $\frac{\mu_o}{\rho_\infty}M\frac{\partial H}{\partial x}$ in (3), represents the component of the magnetic force, known as "Kelvin body force term", per unit volume, in the x-direction and depends on the existence of the magnetic gradient [6], [24]~[26]. The term $\frac{\mu_o}{\rho_\infty c_p}T\frac{\partial M}{\partial T}(u\frac{\partial H}{\partial x}+\upsilon\frac{\partial H}{\partial y})$ in (4), represents the thermal power, per unit volume, due to the magneto-caloric effect.

There are various equations describing the dependence of the magnetization M and some of them are mentioned in [6]~[9], [26]~[28]. For the physical problem under consideration the simplest expression is adopted [9], [26], [28]. This equation is of the form

$$M=\chi H\,, \qquad (7)$$

where χ is the magnetic susceptibility of the fluid and it is assumed to be a constant taking positive or negative values for magnetic fluids exhibiting paramagnetic (χ >0) or diamagnetic (χ <0) behaviour. This ability of changing the sign of magnetization due to the change of the sign of χ, permits us to consider that the magnetic body force term of equation (3) takes the form

$$\pm\frac{\mu_o}{\rho_\infty}\chi H\frac{\partial H}{\partial x}$$

The change of the sign of the magnetic force term denotes that the fluid is attracted or repelled by the application of the magnetic field which is valid for diamagnetic or paramagnetic materials. Attraction or repulsion of the fluid could be possibly achieved artificially by proper construction of magnetic nanoparticles which is plausible for biomedical applications [28], [29].

It is also known that, in the problem under consideration, the development of the boundary layer near the leading edge, where the local Reynolds number Re_x is low or the values of x are small, is dominated by the large viscous forces and the buoyancy force may be neglected. However, further along the plate, or for large values of x, these viscous forces diminish and the buoyancy, due to the temperature difference $\Delta T = T_w - T_\infty$, plays a significant role to the evolution of the boundary layer. Thus, in order to study the influence of the applied magnetic field to the flow field, two separate cases are considered. In the first case (**Case I**), the boundary layer is studied near the leading edge, where the values of x are small and the boundary layer is dominated by the large viscous forces. In the second one (**Case II),** the boundary layer is studied far from the leading edge of the plate where the effects of buoyancy forces increases and the values of x are large.

2.1 Case I

In this case the following dimensionless coordinates of transformation are introduced

$$\xi(x) = \frac{Gr_x}{\mathrm{Re}_x^2} = cx, \quad \eta(x,y) = \mathrm{Re}_x^{1/2}\,\frac{y}{x}, \quad y \geq 0, \quad x > 0, \tag{8}$$

where Gr_x and Re_x are the local Grashof and local Reynolds number given by the expressions

$$Gr_\chi = g\beta(T_w - T_\infty)x^3 / v_\infty^2, \mathrm{Re}_x = xu_\infty / v_\infty, \text{ respectively,} \tag{9}$$

$$c = g\beta(T_w - T_\infty)/u_\infty^2 \text{ is a constant} \tag{10}$$

and ν_∞ is the free stream kinematic viscosity ($\nu_\infty = \frac{\mu_\infty}{\rho_\infty}$). In addition, a reduced stream function $f(\xi,\eta)$ and a dimensionless temperature $\Theta(\xi,\eta)$ are defined as follows

$$f(\xi,\eta) = \Psi(x,y)/(\nu_\infty \mathrm{Re}_x^{1/2}), \Theta(\xi,\eta) = (T - T_\infty)/(T_w - T_\infty) \tag{11}$$

The function $\Psi(x,y)$, is the stream function that ensures that (2) is automatically satisfied, i.e

$$(u,\upsilon) = (\frac{\partial\Psi}{\partial y}, -\frac{\partial\Psi}{\partial x}). \tag{12}$$

In many research works it was considered that the viscosity and the thermal conductivity of the liquid carrier of a magnetic fluid are constants. However, it is known that these physical properties may change significantly with temperature and the flow characteristics substantially change compared with the constant viscosity and the thermal conductivity cases. Thus, a viscosity dependence on temperature T, suggested in [30], is of the form

$$\mu = \frac{\mu_\infty}{1 + \gamma(T - T_\infty)} \tag{13}$$

in which the viscosity μ of the fluid is an inverse linear function of temperature T.

Equation (13) can be rewritten as

$$1/\mu = \alpha(\mathrm{T} - \mathrm{T}_r) \text{ where } \alpha = \gamma/\mu_\infty \text{ and } T_r = T_\infty - 1/\gamma, \tag{14}$$

where both α and T_r are constants the values of which depend on the reference state and γ is a constant connected with the thermal property of the fluid.

The dimensionless temperature Θ can also be written now as

$$\Theta = (T - T_r)/(T_w - T_\infty) + \Theta_r . \tag{15}$$

where the parameter Θ_r is defined as

$$\Theta_r = (T_r - T_\infty)/(T_w - T_\infty) = -1/\gamma(T_w - T_\infty) = const. \tag{16}$$

and its value is determined by the viscosity/temperature characteristics of the fluid and the operating temperature difference $\Delta \mathrm{T} = \mathrm{T}_w - \mathrm{T}_\infty$.

It is worth noting here that, in the case where the temperature difference $\Delta\mathrm{T}=\mathrm{T}_w - T_\infty$ is positive, the viscosity/temperature parameter Θ_r is negative for liquids and positive for gases, since γ has the opposite sign in each of these cases. It has also been reported in [31] and [32] that, for physical reality, Θ_r cannot take values between 0 and 1 and it is $\Theta_r > 1$ for gases and $\Theta_r < 0$ for liquids. This merely reflects the physical property of the viscosity in these two stages of matter either increasing (gases) or decreasing (liquids) with increasing temperature. It is also noted that when Θ_r is large, the viscosity variation in the boundary layer is negligible. On the contrary, as $\Theta_r \to 1^+$ for gases, or $\Theta_r \to 0^-$ for liquids, the viscosity variation becomes increasingly significant. On the other hand, for most liquids the thermal conductivity k is expressed by a linear function of temperature of the form [33]

$$k = k_\infty[1 + s(T - T_\infty)]. \tag{17}$$

where k_∞ is the ambient fluid thermal conductivity and s is a constant depending on the nature of the fluid. This form can also be rewritten as

$$k = k_\infty(1 + S^*\Theta) \tag{18}$$

where

$$S^* = s(T_w - T_\infty) \tag{19}$$

is the thermal/conductivity parameter and its values are determined by the nature of the fluid as well as by the operating temperature difference $\Delta\mathrm{T} = \mathrm{T}_w - \mathrm{T}_\infty$.

The substitution of (1) and (7) ~ (19), into equations (3) and (4), gives the following system

$$f''' - \frac{1}{2} f \frac{\Theta - \Theta_r}{\Theta_r} f'' - \frac{\Theta'}{\Theta - \Theta_r} f'' - \xi(\Theta - \Theta_r)\frac{\Theta}{\Theta_r} = \xi\{\frac{\partial f}{\partial \xi} f'' - f' \frac{\partial f'}{\partial \xi}\}\frac{\Theta - \Theta_r}{\Theta_r}$$

$$\mp RMn \frac{\Theta - \Theta_r}{\Theta_r} \frac{\xi(\xi - ac)}{[(\xi - ac)^2 + (d\sqrt{\xi}\eta - cb)^2]^2}, \tag{20}$$

$$\frac{1}{\Pr}(1+S^{*}\Theta)\ \Theta''+(\frac{1}{2}\mathrm{f}+\frac{1}{\Pr}\mathrm{S}^{*}\Theta^{'})\Theta^{'}=\xi(\mathrm{f}^{'}\frac{\partial\Theta}{\partial\xi}-\frac{\partial \mathrm{f}}{\partial\xi}\Theta^{'})\ . \tag{21}$$

where $\Pr = \frac{\mu_\infty c_p}{k_\infty}$ is the Prandtl number, prime denotes partial differentiation with respect to the variable η, $d = \sqrt{\frac{c\nu_\infty}{u_\infty}}$ and RMn is defined as $RMn = Mn(cb)^2$, with Mn being the magnetic parameter, or magnetic number expressing the ratio of the magnetic forces to the inertia forces (per unit volume) and defined by

$Mn = \frac{M_0 \mathrm{B}_0}{\rho_\infty u_\infty^2}$, where $M_0 = \chi H_0$, $\mathrm{B}_0 = \mu_0 H_0$ and H_0 is the magnetic field intensity at the point $(x_0, y_0, z_0) = (\alpha, 0, z)$, i.e. (equation (1)) $H_0 = H(a,0) = \frac{I}{2\pi}\frac{1}{\sqrt{b^2}}$.

Finally, the boundary conditions (5) and (6) become

$$\eta = 0: \quad f' = 0, \quad f = 0,\ \Theta = 1, \tag{22}$$

$$\eta \to \infty: \quad f' = 1, \quad \Theta = 0. \tag{23}$$

The dimensionless parameters entering into the problem under consideration, described by the system (20) ~ (23) for the **Case I**, are the Prandtl number Pr, the magnetic number Mn, the viscosity/temperature parameter Θ_r and the thermal/conductivity parameter S^*. The system (20)~(21) is a coupled, nonlinear system of partial differential equations of parabolic type, with the unknown functions $f(\xi,\eta)$ (or $f'(\xi,\eta) = u/u_\infty$), $\Theta(\xi,\eta)$, defined in the semi-infinite rectangular domain D={ (ξ,η) / 0< ξ <1, 0$\le \eta \le \eta_\infty$ }, subject to the boundary conditions (22)~(23).

2.2 Case II

For this case it is helpful to define the dimensionless coordinates ξ and ζ by the following expressions [34], [35].

$$\xi(x) = \frac{Gr_x}{\mathrm{Re}_x^2} = cx, \quad \zeta(x,y) = (\frac{g\beta\Delta T}{\nu_\infty^2})^{\frac{1}{4}} \frac{y}{x^{\frac{1}{4}}}, \quad y \ge 0, \quad x > 0. \tag{24}$$

On the other hand, for this case, a reduced stream function $F(\xi,\zeta)$ and a dimensionless temperature $\Theta(\xi,\zeta)$ are defined as follows

$$F(\xi,\zeta) = \Psi(x,y)/[g\beta\Delta T\nu_\infty^2 x^3]^{\frac{1}{4}},\ \Theta(\xi,\zeta) = (T - T_\infty)/(T_w - T_\infty) \tag{25}$$

Following the same procedure, as in **Case I,** the system of equations describing the flow in the region of large values of x, or ξ, is

$$F''' - \frac{3}{4}\frac{\Theta-\Theta_r}{\Theta_r}FF'' - \frac{\Theta'}{\Theta-\Theta_r}F'' + \frac{1}{2}\frac{\Theta-\Theta_r}{\Theta_r}F'^2 - (\Theta-\Theta_r)\frac{\Theta}{\Theta_r} = \xi\{\frac{\partial F}{\partial\xi}F'' - F'\frac{\partial F'}{\partial\xi}\}\frac{\Theta-\Theta_r}{\Theta_r}$$
$$\mp RMn\frac{\Theta-\Theta_r}{\Theta_r}\frac{(\xi - ac)}{[(\xi-ac)^2 + (d\xi^{\frac{1}{4}}\zeta - cb)^2]^2}, \tag{26}$$

$$\frac{1}{\Pr}(1+S^*\Theta)\ \Theta'' + (\frac{3}{4}\mathrm{F} + \frac{1}{\Pr}\mathrm{S}^*\Theta')\Theta' = \xi(\mathrm{F}'\frac{\partial\Theta}{\partial\xi} - \frac{\partial \mathrm{F}}{\partial\xi}\Theta')\,, \tag{27}$$

whereas the boundary conditions of this system are

$$\zeta = 0: \quad \mathrm{F}' = 0, \quad \mathrm{F} = 0, \quad \Theta = 1, \tag{28}$$

$$\zeta \to \infty: \quad \mathrm{F}' = \frac{1}{\sqrt{\xi}}, \quad \Theta = 0. \tag{29}$$

The dimensionless parameters entering into this problem, described by the system (26) ~ (29) for the **Case II**, are the same to those in **Case I**. In this case, however, prime denotes partial differentiation with respect to the variable ζ. The system (26)~(27) is a coupled, nonlinear system of partial differential equations of parabolic type, with the unknown functions $\mathrm{F}(\xi,\zeta)$ (or $\mathrm{F}'(\xi,\zeta)\sqrt{\xi} = \mathrm{u}/\mathrm{u}_\infty$), $\Theta(\xi,\zeta)$, defined in the semi-infinite rectangular domain D = $\{(\xi,\zeta)/ 1 \le \xi < \xi_\infty,\ 0 \le \zeta \le \zeta_\infty\}$, subject to the boundary conditions (28) ~ (29). Also, it is worth noting here that in the case where Mn =0 and S^* = 0, the problem under consideration, for both **Cases**, becomes similar to that studied by Kafoussias et al. [32], for the case of a non magnetic fluid (air or water) and solved by a different numerical technique.

3. Numerical solution technique

3.1 Case I

For this case (small values of ξ), the physical problem is described by the system of non-linear equations (20) ~ (21) subject to the boundary conditions (22) and (23). It is worth reminding here that the dimensionless coordinate ξ, is defined, for both Cases, as $\xi(x) = \frac{Gr_x}{\mathrm{Re}_x^2} = cx$, where $c = g\beta(T_w - T_\infty)/u_\infty^2$ is a constant. In a single experiment, when the fluid and temperature parameters are fixed, ξ may be regarded as a dimensionless distance along the plate from the leading edge, whereas changing the fluid and temperature parameters merely alters the scale of the distance relative to the actual distance x. Near the leading edge, ξ<<1 forced convection dominates. As ξ increases (ξ~1), the fluid moves into the mixed convection regime and subsequently into a free convection dominated flow (ξ>>1).

The system (20)~(21) as well as the system (26)~(27), is of parabolic type and it can be solved by several numerical methods discussed in [35]~[38]. The applied numerical scheme, used to solve the system under consideration, consists in proceeding in the ξ- direction, i.e. calculate unknown profiles at ξ_{i+1} when the same profiles at ξ_i are known. However, as the system is of parabolic type, initial (at ξ=0) velocity and temperature profiles are needed to start the procedure. Thus, the process starts at ξ = 0 where equations (20) and (21) reduce to

$$f''' - \frac{1}{2} f \frac{\Theta - \Theta_r}{\Theta_r} f'' - \frac{\Theta'}{\Theta - \Theta_r} f'' = 0, \tag{30}$$

$$\frac{1}{\Pr}(1 + S^* \Theta)\ \Theta'' + (\frac{1}{2} f \frac{1}{\Pr} S^* \Theta')\Theta' = 0. \tag{31}$$

The boundary conditions for these equations are the same as those of the complete system of equations. The numerical solution of the above system is easily obtained by applying the efficient numerical technique described in detail in [39] and is similar to that presented in [40], [41]. This Crank–Nicolson-type numerical scheme is $O(\Delta\xi^2)$ and $O(\Delta\eta^2)$ and is implemented on a uniform $\xi - \eta$ grid. The steps $\Delta\xi$ (=0.02) and $\Delta\eta$ (=0.05) are experimentally adjusted until the results obtained are not numerically sensitive beyond the desired accuracy. For the numerical applications and for the obtained results, in the work under consideration, the maximum value of ξ (ξ_final) has been chosen to be equal to 0.8 although the numerical solution can proceed, without any difficulty, for larger values of ξ.

3.2 Case II

For the numerical solution of the system (26)~(27) subject to the boundary conditions (28)~(29) a similar numerical procedure as that for the system of **Case I** is applied. In this case, however, to proceed the solution from ξ_i to ξ_{i+1}, initial velocity ($F'(\xi,\zeta) = (u/u_\infty)/\sqrt{\xi}$) and temperature ($\Theta(\xi,\zeta)$) profiles are needed at some initial distance ξ_initial from the leading edge of the plate, that is, for ξ_initial >>0. These initial profiles are obtained from the solution of the system in **Case I**, up to the point ξ_final = ξ_initial =1. Then, the solution of the system in **Case I**, at the point ξ_initial =1, is used as initial conditions to start the solution of the system in **Case II**. Then, the solution proceeds for every ξ >1, up to a final distance from the leading edge of the plate, that is, up to the $\xi = \xi'$_final. To verify the correctness of this choice, solutions of both systems were obtained for the same point $(\bar{x}, \bar{y})$ inside the boundary layer or for the corresponding points $(\xi(\bar{x}), \eta(\bar{x}, \bar{y})) \leftrightarrow (\xi(\bar{x}), \zeta(\bar{x}, \bar{y}))$ and in a region where both solutions could be valid, e.g. for $\xi(\bar{x})$ =1.3. The solution of the system in **Case I** was compared with the solution of the system in **Case II** and both solutions were found to be in excellent agreement. The maximum value of ξ (ξ'_final), for this case, has been chosen to be equal to ξ'_final = 12.0.

4. Results and discussion

In order to study, numerically, the effects of the various parameters of the problem under consideration on the flow field of a magnetic fluid, the following assumptions are adopted for both **Cases I** and **II**. The magnetic fluid is considered to have water as carrier liquid, i.e. it is a water-based magnetic fluid. The temperature T_w of the vertical plate as well as the fluid temperature T_∞ in the free stream are considered to be constants and equal to T_w=353 ^{0}K

($80^0\,C$) and $T_\infty = 293^0$ K ($20^0\,C$), respectively. Thus, the temperature difference $\Delta T = T_w - T_\infty$ is equal to 60^0K. The free stream velocity u_∞ of the magnetic fluid is taken equal to $u_\infty = 0.28$ m/sec and the acceleration due to gravity g = 9.81m/sec^2. Following [5], [19] and [20], it is considered that the representative magnetic fluid at a reference temperature 300^0K ($27^0\,C$) has the following physical properties: The fluid density ρ_∞, at the free steam is ρ_∞=1180 kgr/m^3, the coefficient of the thermal expansion β of the fluid is $\beta = 5.6\cdot10^{-4}\ {}^0\text{K}^{-1}$, the viscosity of the fluid μ_∞ is μ_∞=0.007 Kgr m$^{-1}\cdot$sec^{-1}, the coefficient of thermal diffusion is $a = 1.19\text{x}10^{-7}\ \text{m}^2\,\text{sec}^{-1}$ and the specific heat under constant pressure is c_p = 4180 Jkgr^{-1}. Under these assumptions the free stream kinematics viscosity of the fluid is $\nu_\infty = \dfrac{\mu_\infty}{\rho_\infty}$ = $5.932\text{x}10^{-6}\text{m}^2\,\text{sec}^{-1}$ and the Prandtl number is Pr = 49.832.

On the other hand, following [20] and assuming that the value H_0 of the magnetic field intensity at the point $(x_0, y_0, z_0) = (\alpha, 0, z)$, i.e. $H_0 = H(a,0) = \dfrac{I}{2\pi}\dfrac{1}{\sqrt{b^2}}$, that saturates the magnetic fluid, is about $1.3\text{x}10^6\,\text{A/m}$ and the magnetic susceptibility χ of the liquid carrier (water), at 20 ^{0}C and in the S.I., is $\chi = 9.04\text{x}10^{-6}$, the magnetic parameter Mn takes the value Mn=0.12. Also, the constant c in the transformation $\xi(x) = cx$, defined as $c = g\beta(T_w - T_\infty)/u_\infty^2$ (equation (10)) and the constant d, defined as $d = \sqrt{\dfrac{c\nu_\infty}{u_\infty}}$, take the values c = 3.6624 and d = $8.5084\text{x}10^{-3}$, respectively. The above considerations permit us to adopt values for the above parameters that correspond to a realistic case scenario. For the numerical solution we additionally take into consideration the physically realistic case of negative χ values discussed above. Thus, we also consider the case of χ = -9.04x10^{-6} resulting to Mn=-0.12

For the viscosity/temperature parameter Θ_r, following [30] and [42], it is assumed that it takes the values Θ_r = -20.0 and Θ_r = -0.60. The first value corresponds to the case in which the viscosity of the fluid is not affected by the temperature variations ($\mu = \mu_\infty$=constant), whereas the second one corresponds to the case in which the viscosity of the fluid is sensitive to temperature variations. The thermal/conductivity parameter, $S^* = s(T_w - T_\infty)$, for this choice of the liquid carrier (water) and for the operating temperature difference $\Delta T = T_w - T_\infty = 60^0$K, takes the value S^* = 0.12, [6], [20] and [35]. The value $S^* = s$ =0.0 corresponds to the case in which the thermal conductivity of the fluid is not sensitive to temperature variations ($k = k_\infty$ = constant).

Finally, for both **Cases I** and **II**, it is assumed that the flow is laminar and this is a valid assumption for the cases under consideration. It is known ([35]) that the transitional Reynolds number $R_{x_{tr}}$, at the start of transition, depends partly upon the turbulence in the free stream and greatly upon the surface conditions such as heating or cooling and smoothness or roughness of the plate. Transitional Reynolds number $R_{x_{tr}}$ may be as low as $4x10^5$ or as high as $4x10^6$. If the plate is heated, as in the cases under consideration, the location of natural transition in a gas flow moves upstream, decreasing the value of the transitional Reynolds number, whereas if the plate is cooled, the location of transition moves downstream. The reason is that since μ rises with gas temperature, the velocity gradient near the wall is reduced by heating, distorting the profile to a more unstable shape and for cooling, the converse holds. On the contrary, in liquid flows, μ falls with increasing fluid temperature and the effect is reversed. In the present study, the maximum value of Reynolds number $\mathrm{Re}_x = xu_\infty / v_\infty$, even for ξ'_final = 12.0 or x'_final = 3.28 m, is $1.66x10^5$, i.e, less than $4x10^5$.

4.1 Case I

Near the leading edge the values of x or ξ are small and the boundary layer is dominated by the large viscous forces. In order to study the influence of the action of a localized magnetic field on the flow field in this region, it is assumed that the electric wire is placed at the position (x_0, y_0) = (α , b) = (0.10,-0.20). Taking into account that c = 3.6624, the position of the wire in the transformed and dimensionless coordinate system $O\xi\eta$ is (ξ_0, η_0) = (0.36624,-142.23).

The results obtained by the numerical solution of the system describing the flow near the leading edge, concern the dimensionless quantities of the velocity field $f'(\xi,\eta)$, the temperature field $\Theta(\xi,\eta)$, the skin friction coefficient C_{f_x} and the Nusselt number Nu_x. It is noted here that the skin friction coefficient C_f and the Nusselt number Nu are defined by the expressions

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho_\infty u_\infty^2} \quad (32), \quad \text{and} \quad Nu = xq_w / k(T_w - T_\infty), \text{ respectively,} \qquad (33)$$

where τ_w is given by $\tau_w = (\mu \frac{\partial u}{\partial y})_{y=0}$ (34) and $q_w = -(k\frac{\partial T}{\partial y})_{y=0}$ (35).

Using equations (8) and (11)~(19) in the above expressions, the corresponding dimensionless quantities C_{f_x} and Nu_x can be written as

$$C_{f_x} = C_f \mathrm{Re}_x^{1/2} = \frac{2\Theta_r}{\Theta_r - 1} f''(\xi,0) \quad (36) \qquad \text{and} \qquad Nu_x = Nu \mathrm{Re}_x^{1/2} = -\Theta'(\xi,0) \qquad (37)$$

The presented results are for Mn = 0.00, +0.12, -0.12, Θ_r = -20.0, - 0.60 and S^* = 0.00 and 0.12.

Figures 1 and 2 show the variations of the dimensionless velocity and temperature profiles, respectively, against the dimensionless distance η, normal to the plate and at the place ξ=0.4 along it, i.e., in the region of the magnetic field and just after the electric wire. It is observed that the fluid dimensionless velocity component $f'(\xi,\eta)$, parallel to the plate and inside the boundary layer, increases from zero, on the plate, to its limited value one, at the free stream and it is everywhere greater for positive values of the magnetic parameter Mn.The fluid dimensionless velocity component $f'(\xi,\eta)$ is also greater for Θ_r = - 0.60 than that for Θ_r = -20.0. Hence, the velocity profile is influenced by the presence of the applied magnetic field and by the viscosity/temperature parameter Θ_r. The obtained numerical results, however, showed that the velocity profile is not affected, appreciably, by the thermal/conductivity parameter S^*. On the other hand, the temperature profile $\Theta(\xi,\eta)$ (Fig.2) is not affected, appreciably, by the magnetic parameter Mn but increases as the thermal/conductivity parameter S^* increases from 0.00 to 0.12 or as the viscosity/temperature parameter Θ_r decreases from -0.60 to -20.0

Fig.3 shows the variations of the dimensionless skin friction coefficient C_{f_x}, against the dimensionless distance ξ, for S^* = 0.00, Θ_r = -0.60, -20.0 and for different values of the magnetic number Mn. The corresponding variations for the dimensionless heat transfer coefficient (Nusselt number) Nu_x, are shown in Fig.4. Both quantities increase almost linearly with ξ in the absence of a magnetic field (Mn = 0.0) presenting, however, a different way of variation in the presence of it. For positive values of the magnetic parameter Mn and near the leading edge of the plate (0< ξ < ~ 0.48), the values of C_{f_x} as well as of Nu_x are greater than those for zero on negative values of Mn. However, the opposite is true for large values of ξ (ξ > ~ 0.48). On the other hand, the values of C_{f_x}, for every value of the magnetic parameter Mn, are greater when Θ_r = -20.0 than those when Θ_r = -0.60. The opposite is true, though, for the values of Nu_x. It is reminded that when $|\Theta_r|$ is large, the viscosity variation in the boundary layer is negligible but as $\Theta_r \rightarrow 0^-$, the viscosity variation becomes increasingly significant. Thus, from Figs.3 and 4 it is concluded that the effect of increasing the sensitivity of viscosity to temperature, through the parameter Θ_r, is different for C_{f_x} and Nu_x. The skin friction coefficient is everywhere increased as $|\Theta_r|$ increases, whereas the Nusselt number decreases as $|\Theta_r|$ increases.

Finally, the numerical investigation of the problem under consideration showed that the skin friction coefficient is not influenced, appreciably, by the variation of the thermal/conductivity parameter S^*. On the contrary, the Nusselt number varies appreciably with S^*. These variations for C_{f_x} and Nu_x, are shown in Figs. 5 and 6, respectively, for different values of Mn and for $\Theta_r = -20.0$. It is worth noting that the values of Nu_x, for every value of the magnetic parameter Mn, are greater when S^* = 0.00 than those when S^* = 0.12.

4.2 Case II

In this case the boundary layer is studied far from the leading edge of the plate where the effects of buoyancy forces increase and the values of x or ξ are large (8.0 $< \xi <$ 12.0) . The electric wire is placed now at the position (x_0, y_0) = (α , b) = (2.50,-0.20) or in the transformed and dimensionless coordinate system $O\xi\zeta$ at (ξ_0, ζ_0) = (9.16,-78.26). The dimensionless quantities C_{f_x} and Nu_x, in this case, can be written, (transformations of (24) and (25)), as

$$C_{f_x} = C_f / (d\xi^{\frac{1}{4}}) = \frac{2\Theta_r}{\Theta_r - 1} F''(\xi, 0) \quad (38) \quad \text{and} \qquad Nu_x = Nud / \xi^{\frac{3}{4}} = -\Theta'(\xi, 0), \qquad (39)$$

where C_f and Nu are defined by (32) and (33), respectively, and the constant d is $d = \sqrt{\frac{c\nu_\infty}{u_\infty}}$.

The obtained numerical results concern the dimensionless "velocity" field $F'(\xi,\zeta) = (u/u_\infty)/\sqrt{\xi}$, the temperature field $\Theta(\xi,\zeta)$, the skin friction coefficient C_{f_x} and the Nusselt number Nu_x and are presented in Figures 1~8 for the same values of the dimensionless parameters Mn, Θ_r and S^* as in **Case I**. The variation of the dimensionless profiles $F'(\xi,\zeta)$ against ζ and at the distance ξ = 8.0, in the region of the magnetic field and just before the electric wire, are presented in Fig.1 and 2. Figure 1 shows three representative profiles for Θ_r = -20.0, S^* = 0.00 and Mn = 0.00, +0.12 and -0.12. It is reminded here that the limited value of $F'(\xi,\zeta)$ as $\zeta \to \zeta_\infty$ (ζ_∞ = 10.0), is not 1 as in **Case I** but $1/\sqrt{\xi}$ (boundary condition (29)). It is observed that, as in the case of forced convection flow regime, $F'(\xi,\zeta)$ is always greater for positive value of the magnetic parameter Mn than those for zero or negative values. However, the influence of Mn on $F'(\xi,\zeta)$ is more evident in the free convection flow regime. It is also worth noting that the velocity profiles in the case under discussion (large values of ξ) are of different shape than those of **Case I**. This is due to the fact that the fluid flow now is dominated by the free convection currents and not by the viscous forces as it happens in **Case I**.

The influence of the viscosity/temperature parameter Θ_r as well as of the thermal conductivity/parameter S^*, on the velocity field, is shown in Fig.2. It is concluded that, as in **Case I**, $F'(\xi,\zeta)$ is more sensitive in variations of Θ_r than that of S^*. The dependence of the temperature profile $\Theta(\xi,\zeta)$ on the magnetic parameter Mn, on the viscosity/temperature parameter Θ_r and on the thermal/conductivity parameter S^*, is shown in Figs. 3 and 4, respectively. The influence of these parameters on the temperature field is similar to that on the velocity field.

The variations of the skin friction coefficient C_{f_x}, against ξ, for a positive and negative value of the magnetic parameter Mn as well as for $Mn = 0.0$ and for different values of the viscosity/temperature parameter Θ_r, are presented in Fig.5. The corresponding variations of the dimensionless heat transfer coefficient Nu_x are presented in Fig. 6. Both quantities decrease almost linearly with ξ in the absence of a magnetic field (Mn = 0.0). However, in the presence of the magnetic field, both quantities are influenced by its presence and especially in the region where the magnetic source is located ($\xi_0 = 9.156$). It is remarkable that when Mn = +0.12, C_{f_x} as well as Nu_x increases rapidly with the dimensionless distance ξ from the leading edge of the vertical plate, taking their maximum value in the region where the wire is placed ($\xi \simeq 8.8$). A little far downstream, a corresponding decrement takes place up to the point where they take their minimum value. After that point, C_{f_x} and Nu_x are increased but their values remain lower than the corresponding ones in the absence of the magnetic field. The opposite happens when Mn = -0.12. It is also remarkable that when $\text{Mn} = -0.12$, both C_{f_x} and Nu_x take their minimum value for a value of ξ, i.e. just before the point $\xi = \xi_0$ but after the point where they take their maximum value for $\text{Mn} = 0.12$. Finally, it should be noted that, as in **Case I**, the values of C_{f_x}, for every value of the magnetic parameter Mn, are greater when Θ_r = -20.0 than those when Θ_r = -0.60. The opposite, however, is true for the values of Nu_x.

The variation of the above mentioned quantities C_{f_x} and Nu_x for a positive and negative value of the magnetic parameter Mn as well as for $Mn = 0.0$ and for different values of the thermal/conductivity parameter S^*, are presented in Figs. 7 and 8, respectively. It is concluded that for every value of the magnetic parameter Mn the values of the skin friction coefficient C_{f_x} increase as the values of the thermal/conductivity parameter S^* increases from 0.00 to 0.12 whereas the values of the dimensionless heat transfer coefficient Nu_x decreases. It is worth noting however, that Nu_x is much more sensitive in variations of the values of the thermal/conductivity parameter S^* than C_{f_x} and this is true for every value of the magnetic parameter Mn.

5. Conclusions

The important results, for **both Cases**, of the problem under consideration are summarized as follows: The dimensionless skin friction coefficient C_{f_x}, as well as the dimensionless heat transfer coefficient Nu_x, is influenced by the presence of the magnetic field and especially in the region where the magnetic source is located. This influence is more evident for large values of the dimensionless distance ξ or in the free convection flow region. The effect of increasing the sensitivity of viscosity on temperature, through the parameter Θ_r, is different for C_{f_x} and Nu_x.

The skin friction coefficient is everywhere decreased as Θ_r increases, whereas the Nusselt number increases as Θ_r increases. In the forced convection flow region as well as in the free convection flow region, the velocity profile increases as the magnetic number Mn increases. This increment is more evident for large values of the dimensionless distance ξ or in the free convection flow region. On the other hand, the influence of the magnetic number Mn is almost negligible on temperature profiles. The effect of increasing the sensitivity of viscosity on temperature, through the parameter Θ_r, is different for the velocity field and temperature field. The velocity of the fluid is everywhere increased as Θ_r increases whereas the temperature of the fluid decreases as Θ_r increases. The effect of increasing the sensitivity of thermal conductivity on temperature, through the parameter S^*, is the same for the velocity field as well as for the temperature field. Both quantities are increased as the thermal/conductivity parameter S^* increases. However, the influence of this parameter is more evident on temperature profiles.
When the magnetic fluid exhibits diamagnetic behavior (χ<0 or Mn<0), the effect of the magnetic number Mn on the flow field, is different, qualitatively and quantitatively, with respect to that in the case of a paramagnetic behavior (χ>0 or Mn>0). The velocity field as well as the skin friction coefficient is not influenced, appreciably, by the variation of the thermal/conductivity parameter S^*. The temperature profile as well as the Nusselt number varies with S^*. The influence of the thermal/conductivity parameter S^* on Nusselt number though, is more evident than the corresponding one on the temperature profile. The numerical study of the problem under consideration showed that the magnetic fluid flow is appreciably influenced by the viscosity/temperature parameter Θ_r as well as by the thermal/conductivity parameter S^* and that to predict more accurate results the variable viscosity and thermal conductivity have to be taken into consideration in a magnetic fluid flow.

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Figures Captions

Case I

Fig.1: Variations of the dimensionless velocity profile $f'(\xi,\eta)$, with the dimensionless distance η, for different values of the magnetic number Mn and the viscosity/temperature parameter Θ_r.

Fig.2: Variations of the dimensionless temperature profile $\Theta(\xi,\eta)$ with the dimensionless distance η, for different values of the viscosity/temperature parameter Θ_r and the thermal/conductivity parameter S^*.

Fig.3: Variations of the dimensionless skin friction coefficient C_{f_x}, with the dimensionless distance ξ, for different values of the magnetic number Mn and the viscosity/temperature parameter Θ_r.

Fig.4: Variations of the dimensionless Nusselt number Nu_x, with the dimensionless distance ξ, for different values of the magnetic number Mn and viscosity/temperature parameter Θ_r.

Fig.5: Variations of the dimensionless skin friction coefficient C_{f_x}, with the dimensionless distance ξ, for different values of the magnetic number Mn and the thermal/conductivity parameter S^*.

Fig.6: Variations of the dimensionless Nusselt number Nu_x, with the dimensionless distance ξ, for different values of the magnetic number Mn and the thermal/conductivity parameter S^*.

Case II

Fig.1: Variations of the dimensionless profiles $F'(\xi,\zeta)$ against ζ, for different values of the magnetic number Mn.

Fig.2: Variations of the dimensionless profiles $F'(\xi,\zeta)$ against ζ, for different values of the viscosity/temperature parameter Θ_r and the thermal/conductivity parameter S^*.

Fig.3: Variations of the dimensionless temperature profile $\Theta(\xi,\zeta)$ against ζ, for different values of the magnetic number Mn.

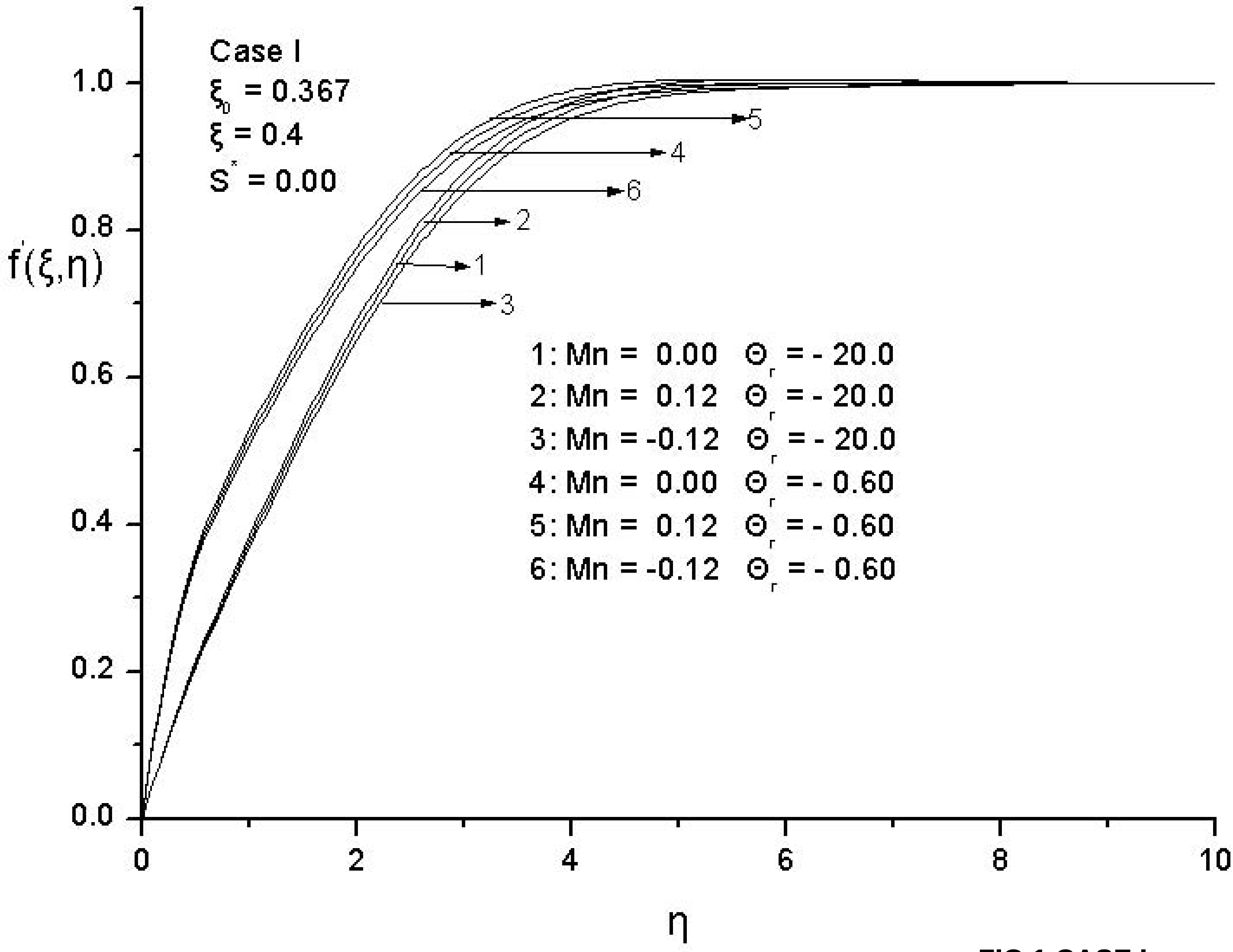
Fig.4: Variations of the dimensionless temperature profile $\Theta(\xi,\zeta)$ against ζ, for different values of the viscosity/temperature parameter Θ_r and the thermal/conductivity parameter S^*.

Fig.5: Variations of the dimensionless skin friction coefficient C_{f_x}, with the dimensionless distance ξ, for different values of the magnetic number Mn and the viscosity/temperature parameter Θ_r.

Fig.6: Variations of the dimensionless Nusselt number Nu_x, with the dimensionless distance ξ, for different values of the magnetic number Mn and viscosity/temperature parameter Θ_r.

Fig.7: Variations of the dimensionless skin friction coefficient C_{f_x}, with the dimensionless distance ξ, for different values of the magnetic number Mn and the thermal/conductivity parameter S^*.

Fig.8: Variations of the dimensionless Nusselt number Nu_x, with the dimensionless distance ξ, for different values of the magnetic number Mn and the thermal/conductivity parameter S^*.



Case I
ξ0 = 0.367
ξ = 0.4
S* = 0.00
f'(ξ,η)
η
1: Mn = 0.00 Θr = - 20.0
2: Mn = 0.12 Θr = - 20.0
3: Mn = -0.12 Θr = - 20.0
4: Mn = 0.00 Θr = - 0.60
5: Mn = 0.12 Θr = - 0.60
6: Mn = -0.12 Θr = - 0.60
1
2
3
4
5
6
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10

FIG 1 CASE I

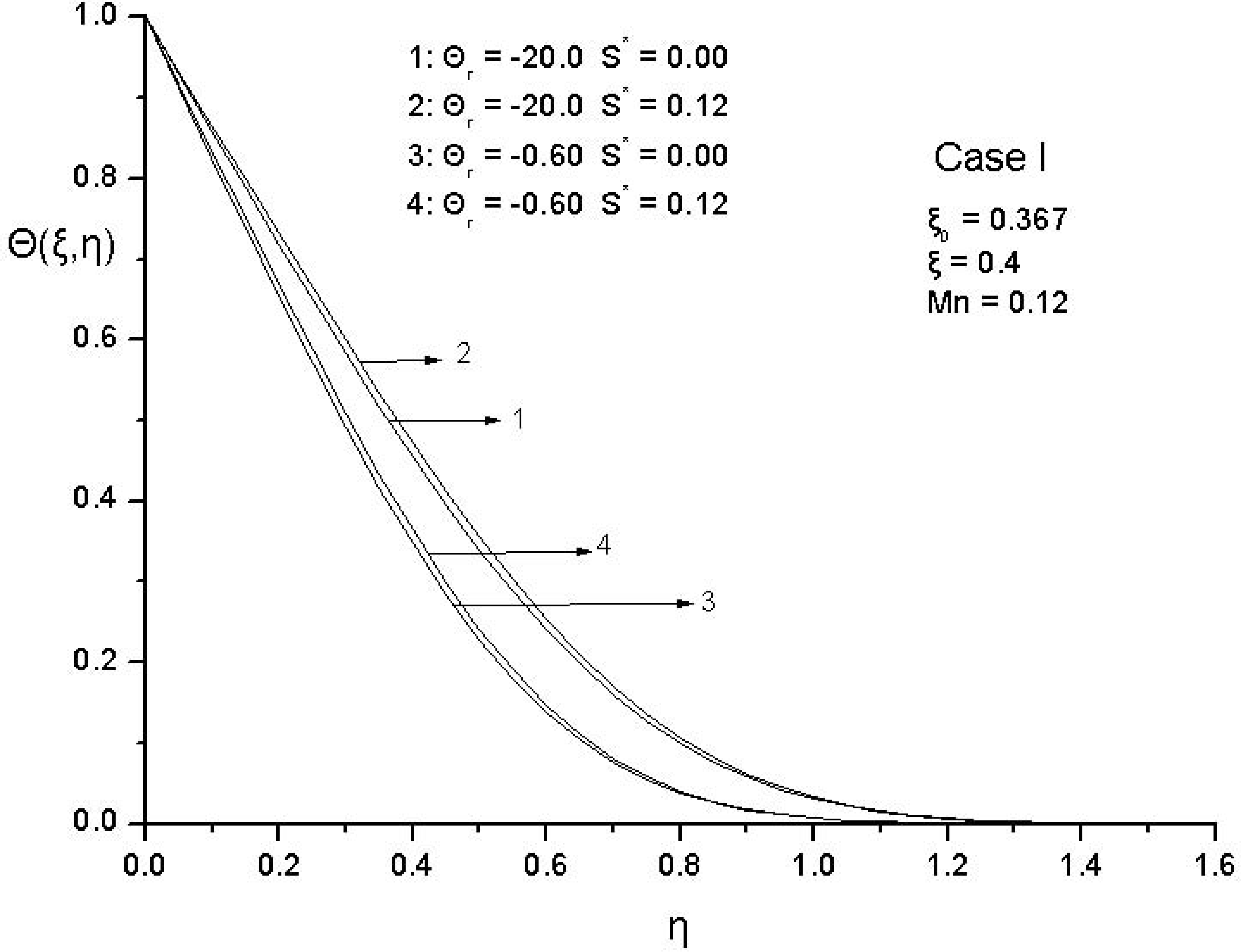

1: Θr = -20.0 S* = 0.00
2: Θr = -20.0 S* = 0.12
3: Θr = -0.60 S* = 0.00
4: Θr = -0.60 S* = 0.12
Case I
ξ0 = 0.367
ξ = 0.4
Mn = 0.12
Θ(ξ,η)
η
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1
2
3
4

FIG 2 CASE I

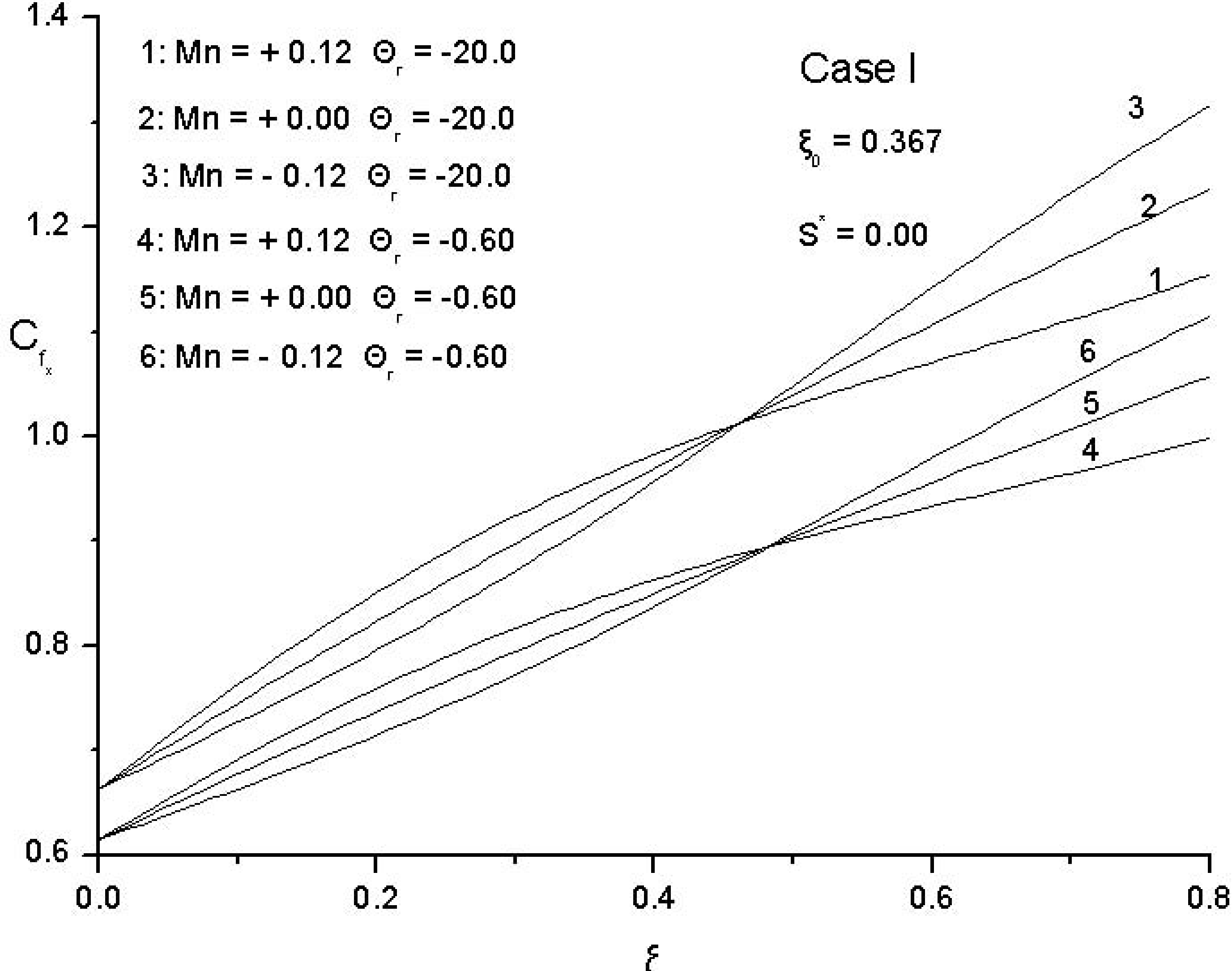

1: Mn = + 0.12 Θr = -20.0
2: Mn = + 0.00 Θr = -20.0
3: Mn = - 0.12 Θr = -20.0
4: Mn = + 0.12 Θr = -0.60
5: Mn = + 0.00 Θr = -0.60
6: Mn = - 0.12 Θr = -0.60
Case I
ξ0 = 0.367
S* = 0.00
Cfx
ξ
1.4
1.2
1.0
0.8
0.6
0.0
0.2
0.4
0.6
0.8
1
2
3
4
5
6

FIG 3 CASE I

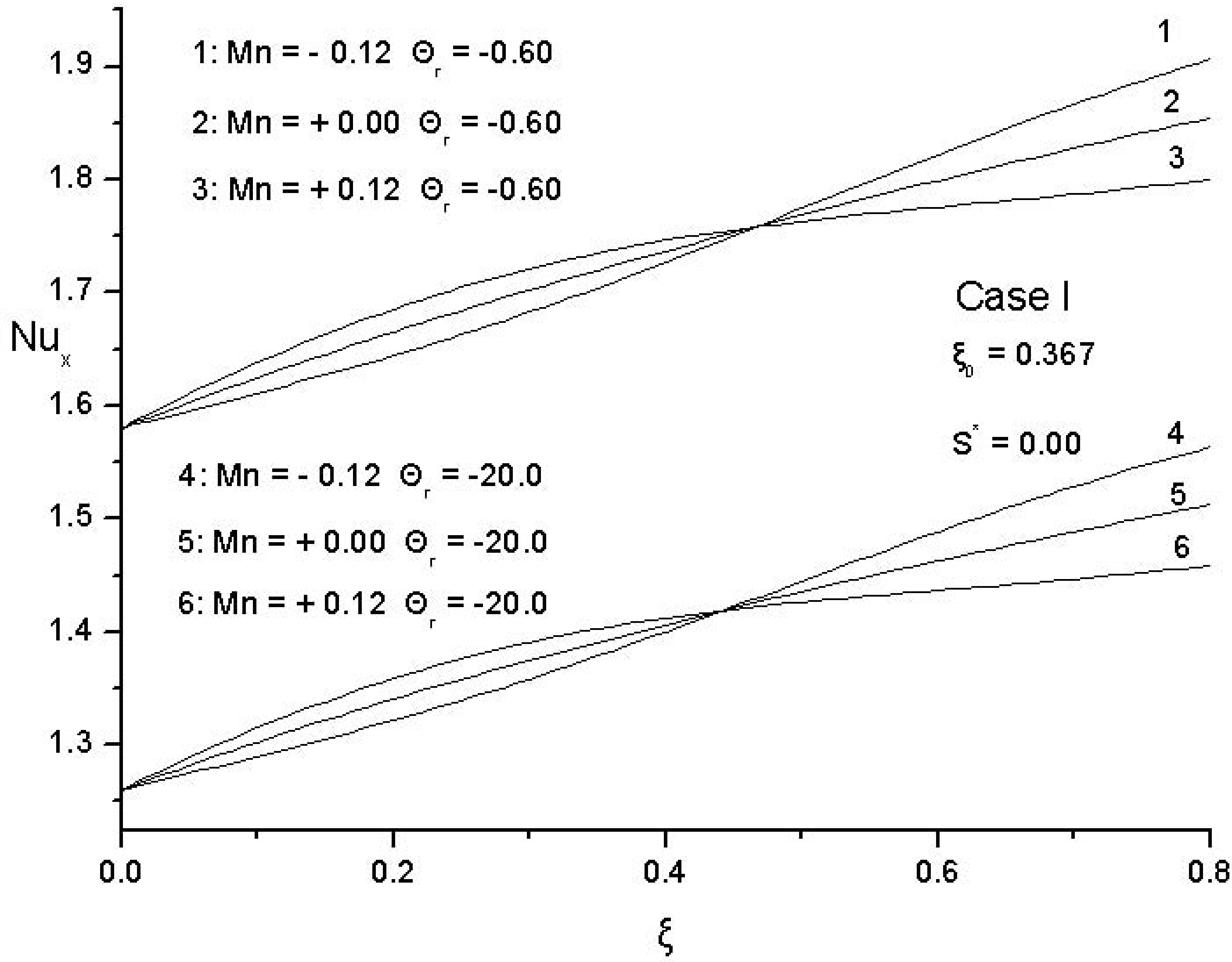
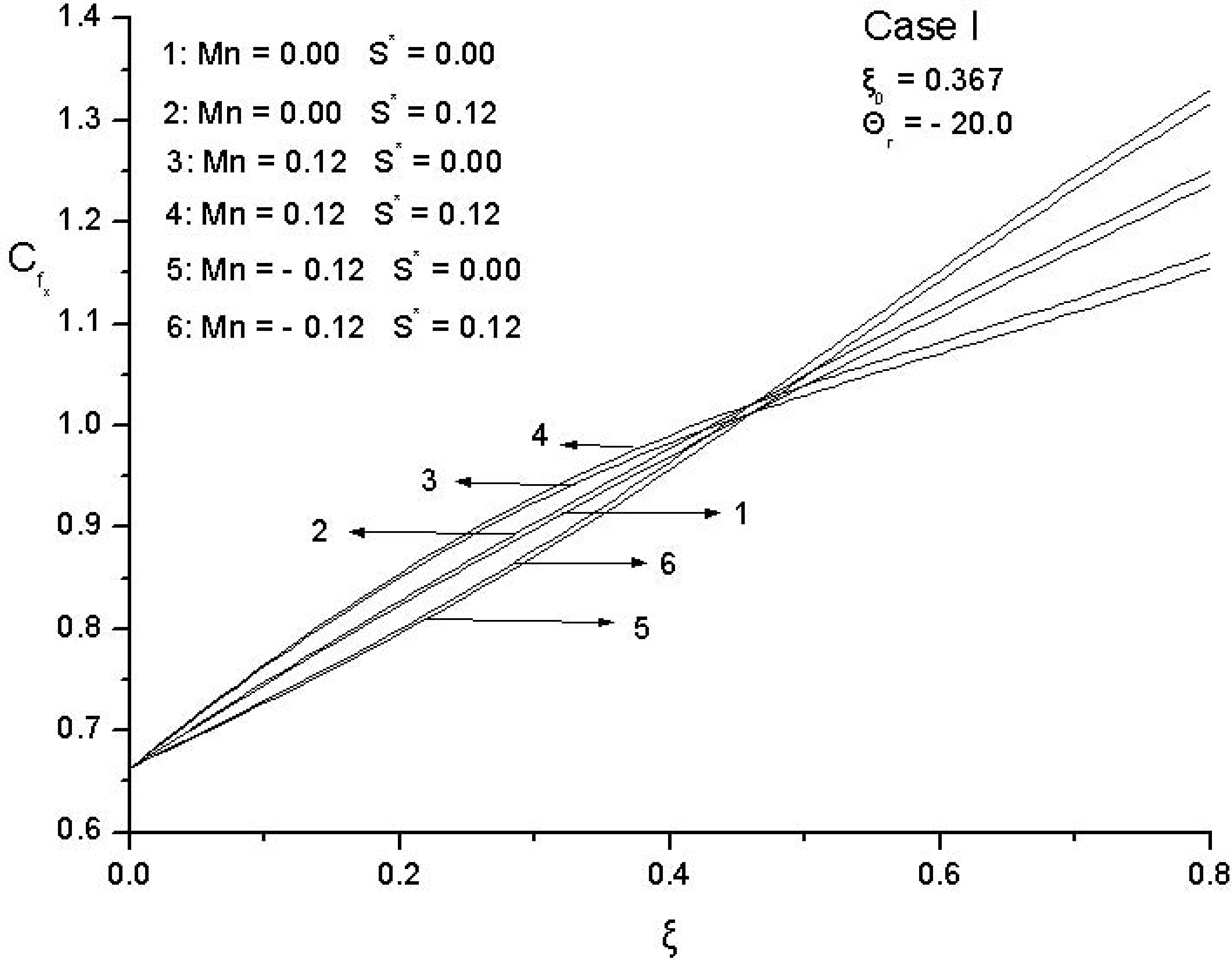

1: Mn = - 0.12 Θr = -0.60
2: Mn = + 0.00 Θr = -0.60
3: Mn = + 0.12 Θr = -0.60
4: Mn = - 0.12 Θr = -20.0
5: Mn = + 0.00 Θr = -20.0
6: Mn = + 0.12 Θr = -20.0
Case I
ξ0 = 0.367
S* = 0.00
Nux
ξ
1
2
3
4
5
6
1.3
1.4
1.5
1.6
1.7
1.8
1.9
0.0
0.2
0.4
0.6
0.8

FIG 4 CASE I



Case I
ξ0 = 0.367
Θr = - 20.0
1: Mn = 0.00 S* = 0.00
2: Mn = 0.00 S* = 0.12
3: Mn = 0.12 S* = 0.00
4: Mn = 0.12 S* = 0.12
5: Mn = - 0.12 S* = 0.00
6: Mn = - 0.12 S* = 0.12
Cfx
ξ
0.6
0.7
0.8
0.9
1.0
1.1
1.2
1.3
1.4
0.0
0.2
0.4
0.6
0.8
1
2
3
4
5
6

FIG 5 CASE I

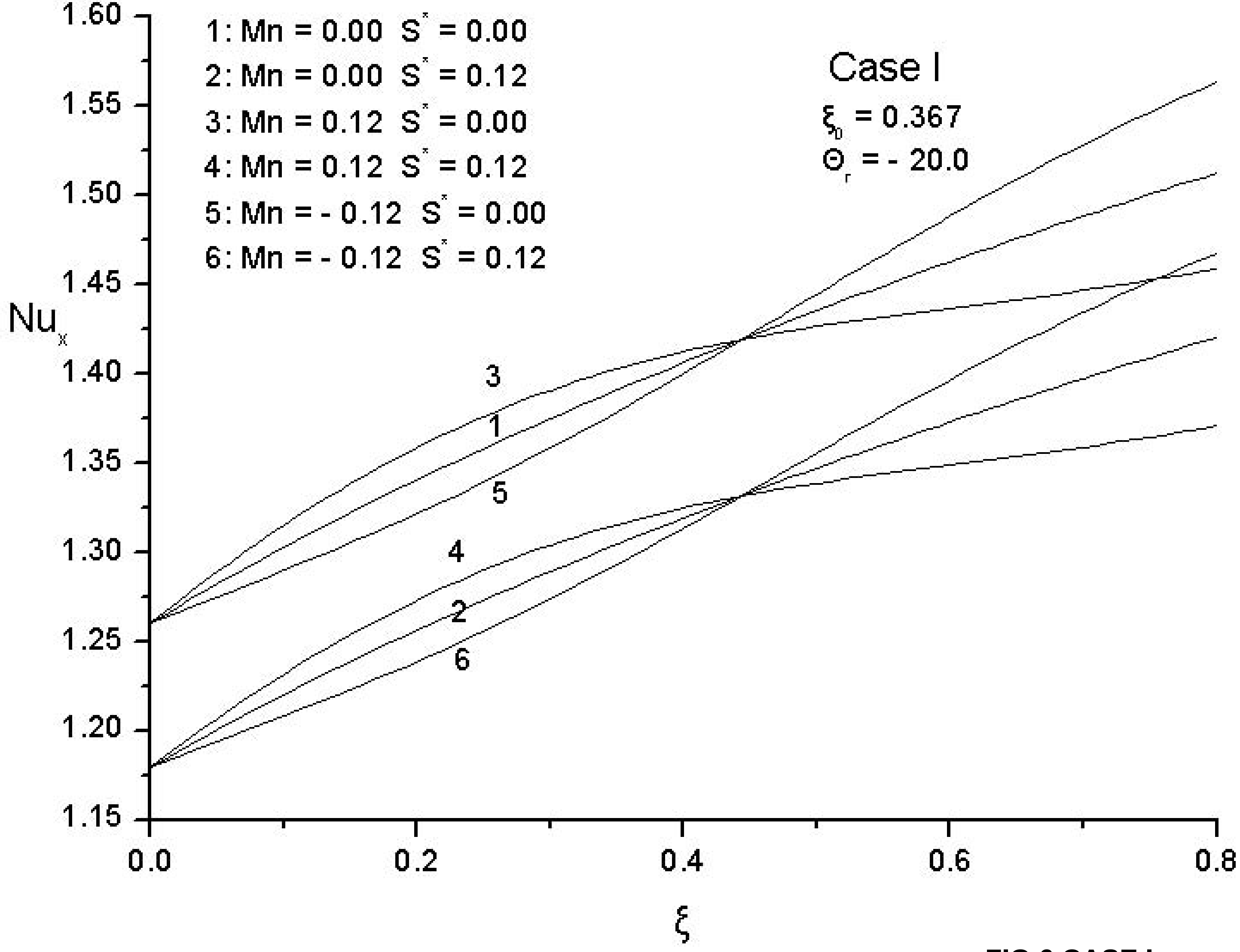

1: Mn = 0.00 S* = 0.00
2: Mn = 0.00 S* = 0.12
3: Mn = 0.12 S* = 0.00
4: Mn = 0.12 S* = 0.12
5: Mn = - 0.12 S* = 0.00
6: Mn = - 0.12 S* = 0.12
Case I
ξ0 = 0.367
Θr = - 20.0
Nux
ξ
1.60
1.55
1.50
1.45
1.40
1.35
1.30
1.25
1.20
1.15
0.0
0.2
0.4
0.6
0.8
3
1
5
4
2
6

FIG 6 CASE I

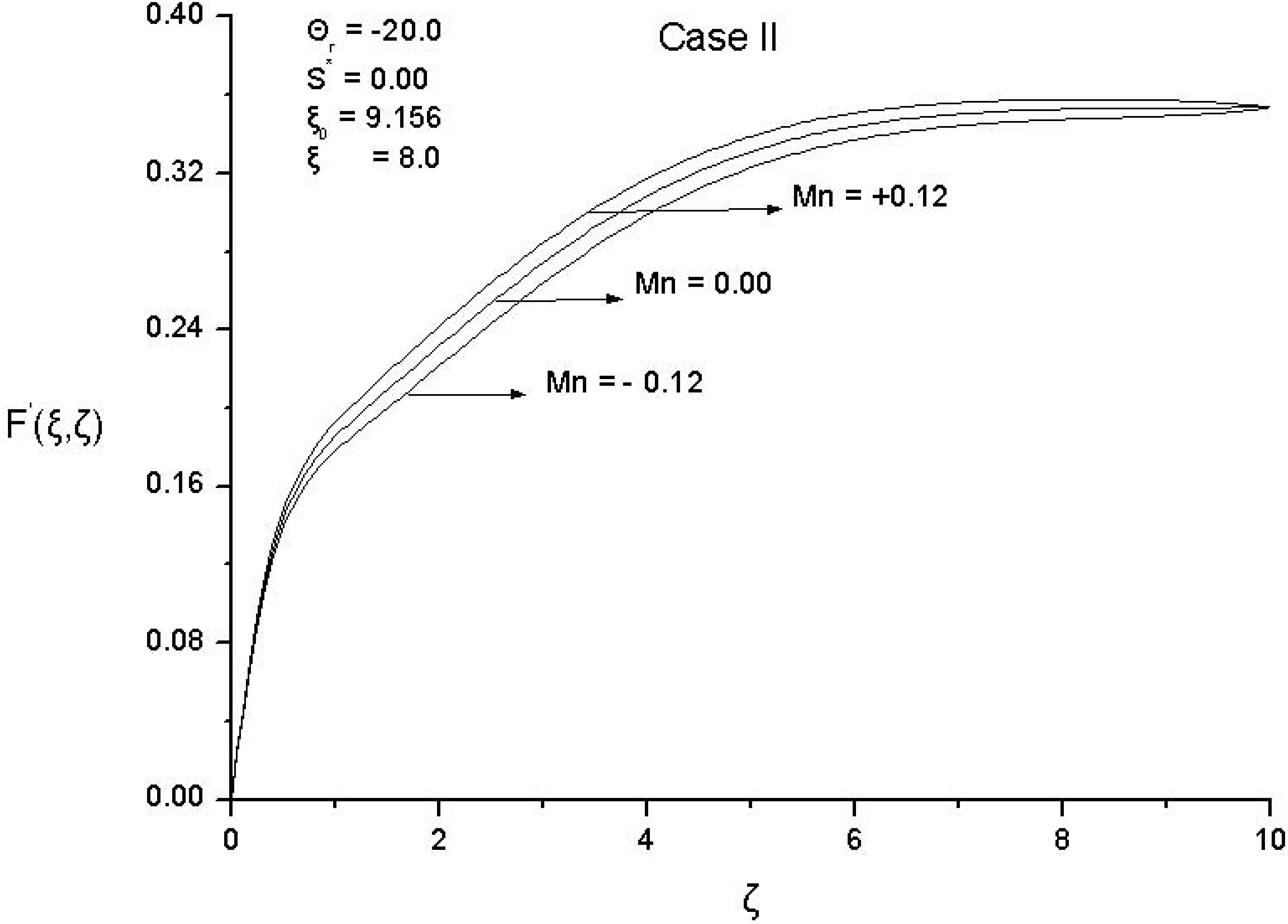
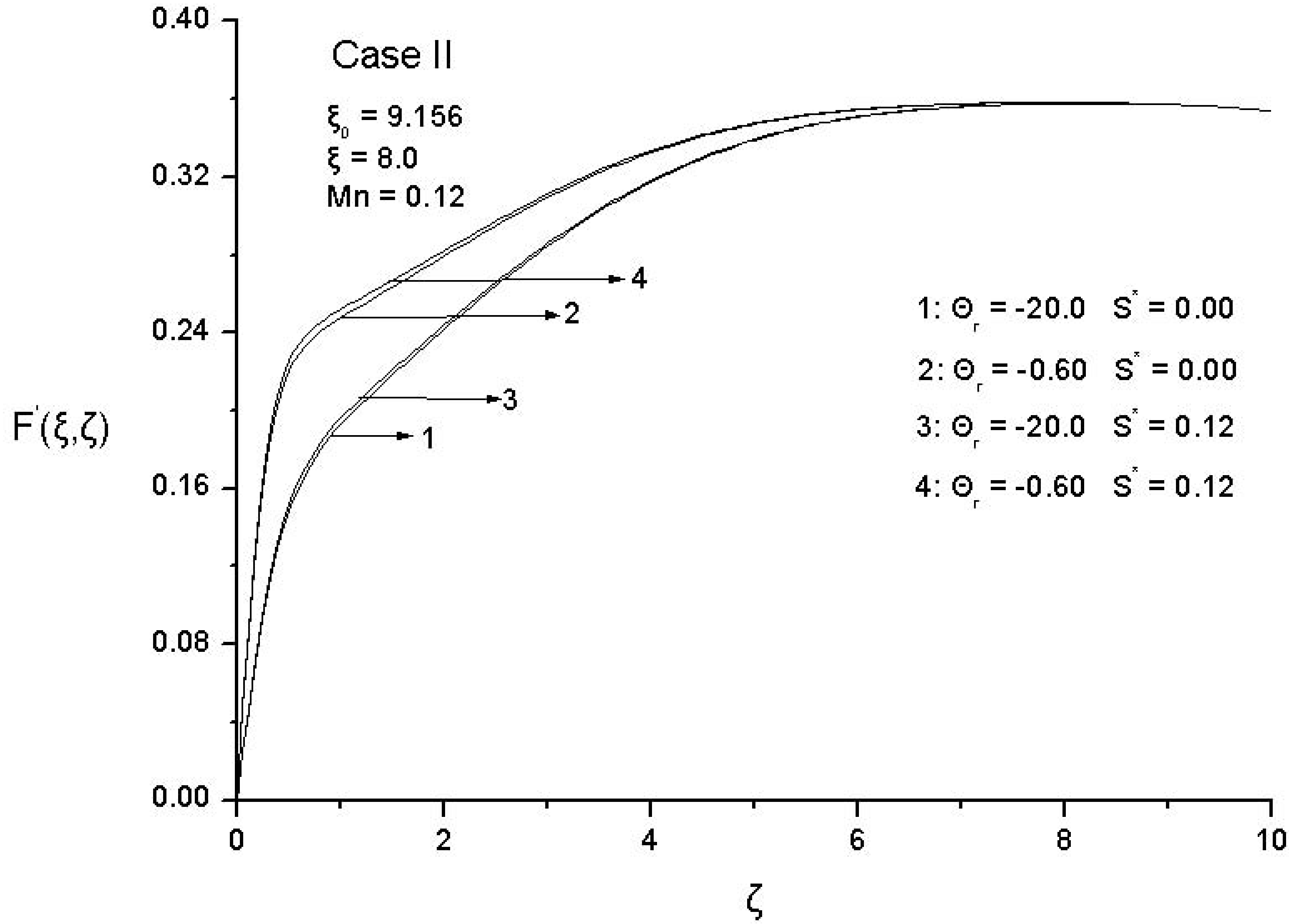

0.40
0.32
0.24
0.16
0.08
0.00
0
2
4
6
8
10
Θr = -20.0
S* = 0.00
ξ0 = 9.156
ξ = 8.0
Case II
Mn = +0.12
Mn = 0.00
Mn = - 0.12
F'(ξ,ζ)
ζ

FIG 1 CASE II



Case II
ξ0 = 9.156
ξ = 8.0
Mn = 0.12
1: Θr = -20.0 S* = 0.00
2: Θr = -0.60 S* = 0.00
3: Θr = -20.0 S* = 0.12
4: Θr = -0.60 S* = 0.12
1
2
3
4
F'(ξ,ζ)
ζ
0.00
0.08
0.16
0.24
0.32
0.40
0
2
4
6
8
10

FIG 2 CASE II

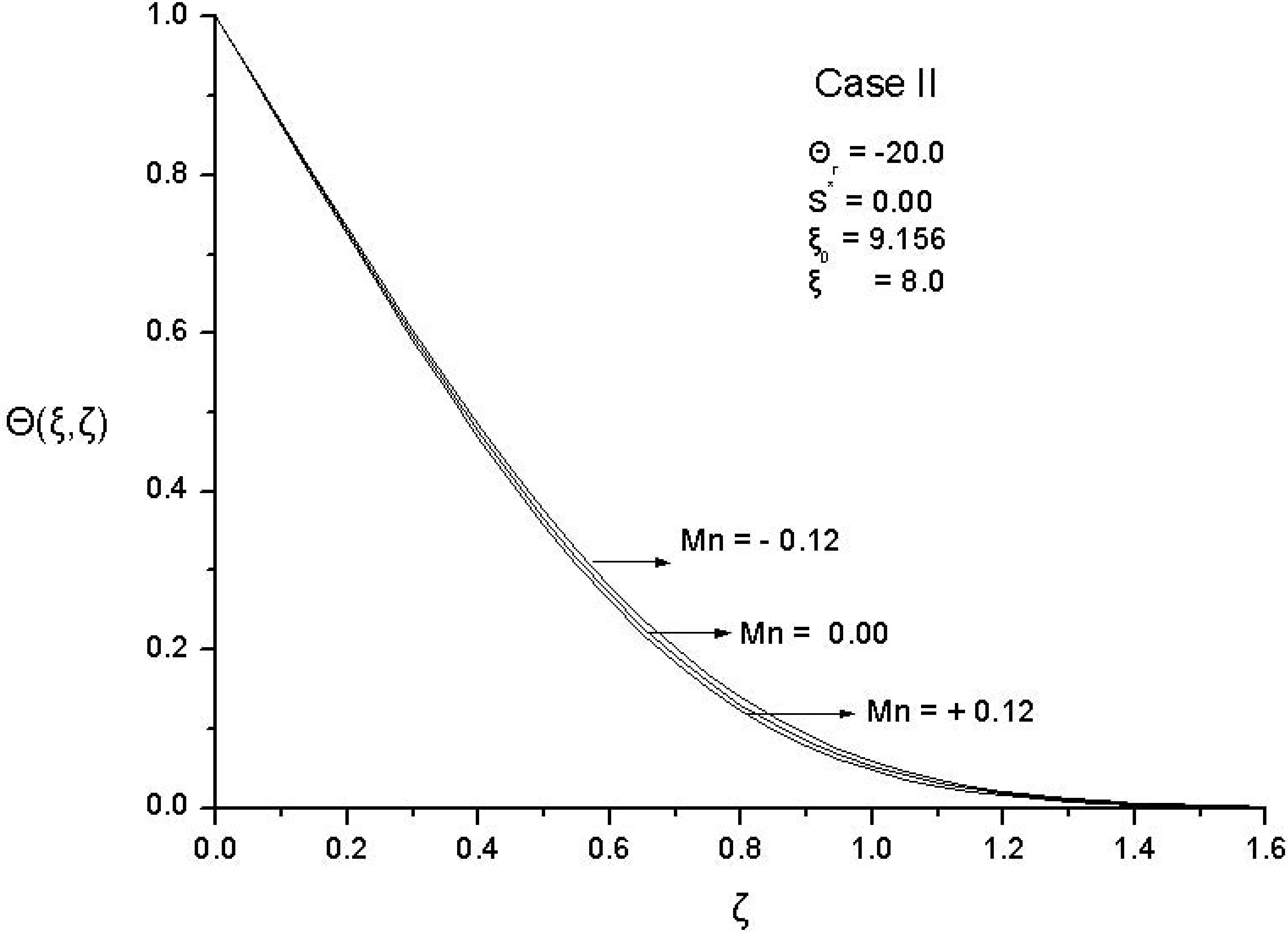
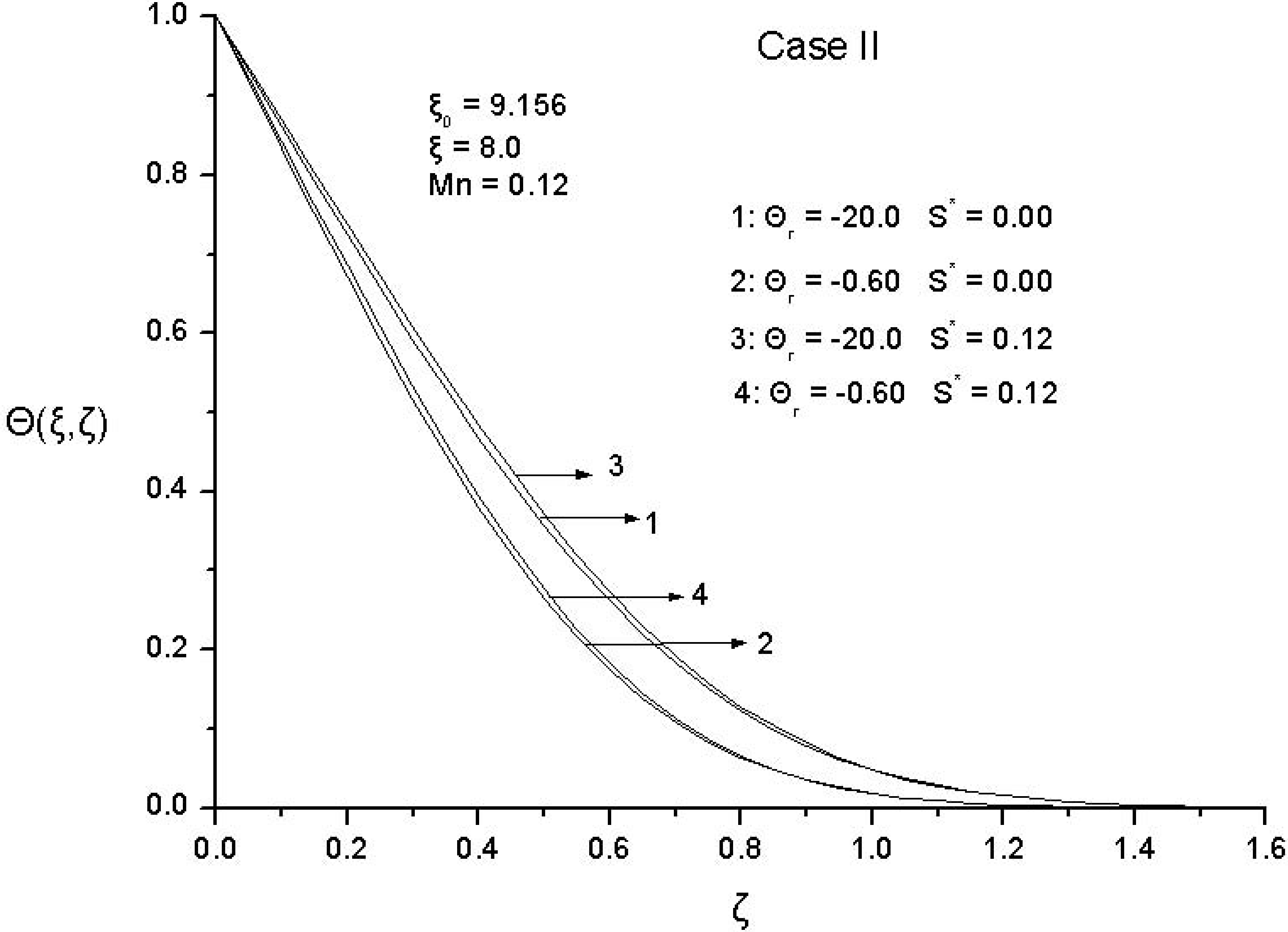

Case II
Θ_r = -20.0
S^* = 0.00
ξ_0 = 9.156
ξ = 8.0
Mn = - 0.12
Mn = 0.00
Mn = + 0.12
$\Theta(\xi,\zeta)$
ζ
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6

FIG 3 CASE II



Case II
ξ0 = 9.156
ξ = 8.0
Mn = 0.12
1: Θr = -20.0 S* = 0.00
2: Θr = -0.60 S* = 0.00
3: Θr = -20.0 S* = 0.12
4: Θr = -0.60 S* = 0.12
Θ(ξ,ζ)
ζ
3
1
4
2
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6

FIG 4 CASE II

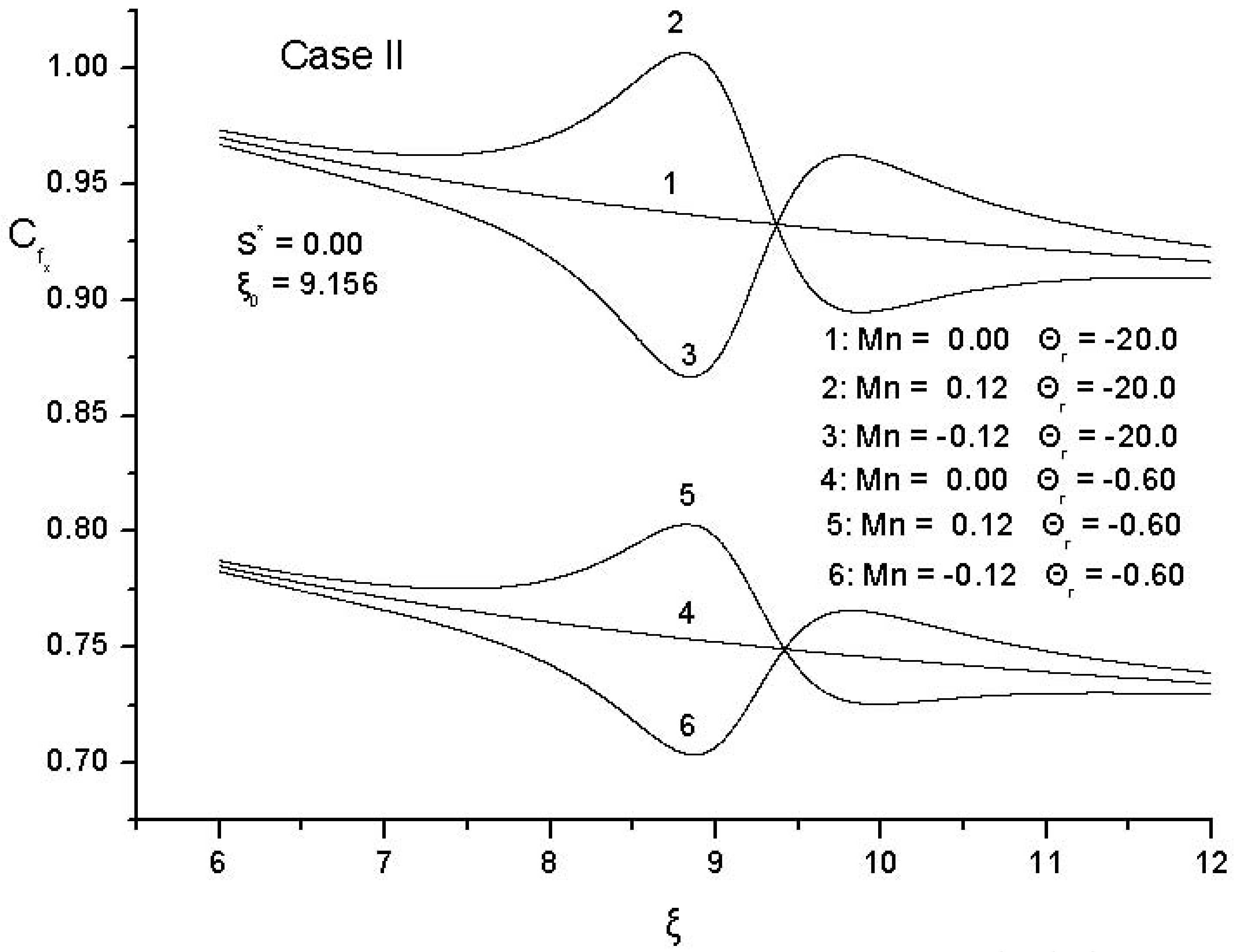
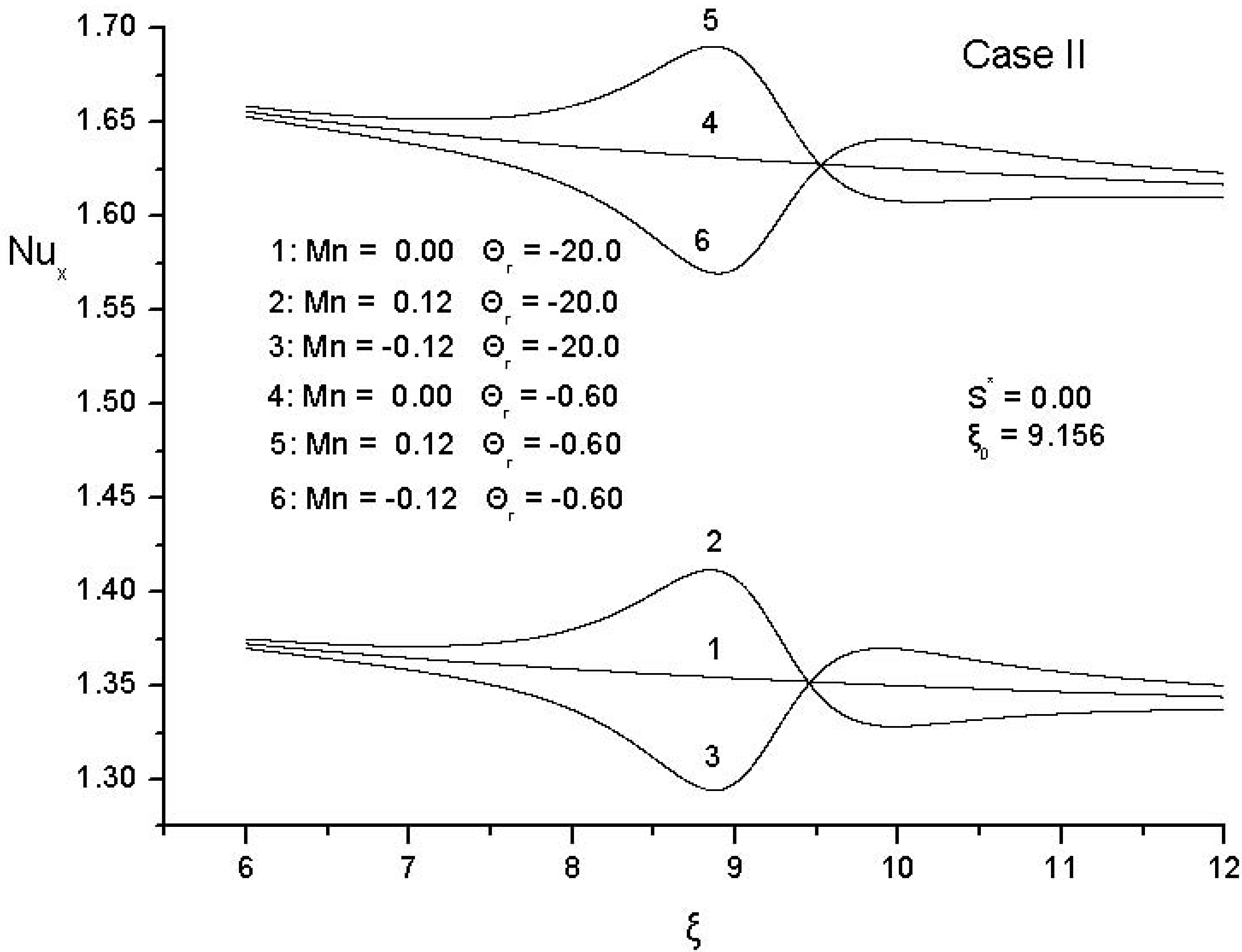
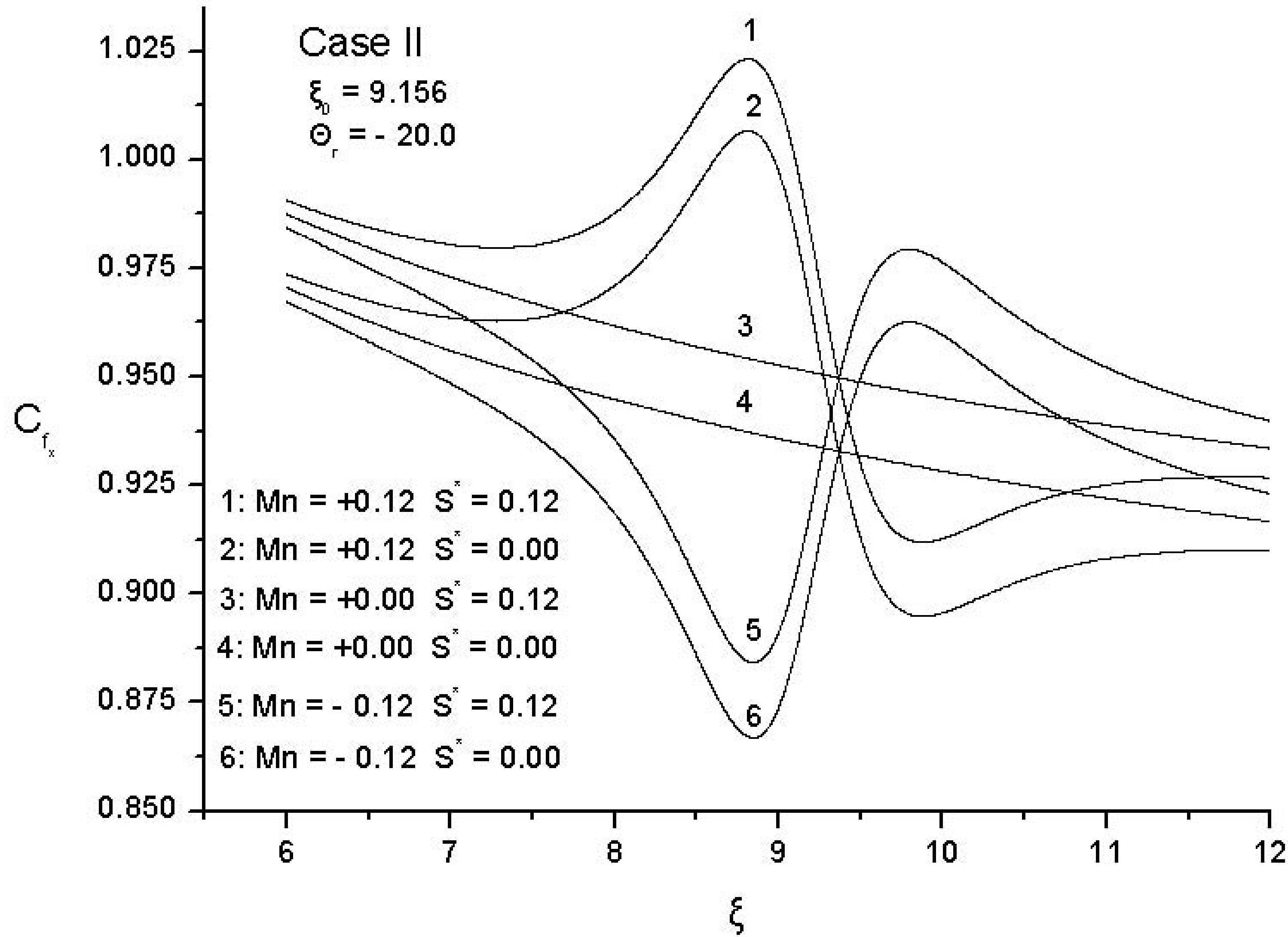

Case II
2
1
3
5
4
6
S* = 0.00
ξ0 = 9.156
1: Mn = 0.00 Θr = -20.0
2: Mn = 0.12 Θr = -20.0
3: Mn = -0.12 Θr = -20.0
4: Mn = 0.00 Θr = -0.60
5: Mn = 0.12 Θr = -0.60
6: Mn = -0.12 Θr = -0.60
Cfx
1.00
0.95
0.90
0.85
0.80
0.75
0.70
6
7
8
9
10
11
12
ξ

FIG 5 CASE II



1.70
1.65
1.60
1.55
1.50
1.45
1.40
1.35
1.30
Nu_x
6
7
8
9
10
11
12
ξ
Case II
1: Mn = 0.00 Θr = -20.0
2: Mn = 0.12 Θr = -20.0
3: Mn = -0.12 Θr = -20.0
4: Mn = 0.00 Θr = -0.60
5: Mn = 0.12 Θr = -0.60
6: Mn = -0.12 Θr = -0.60
S* = 0.00
ξ0 = 9.156
1
2
3
4
5
6

FIG 6 CASE II



Case II
ξ0 = 9.156
Θr = - 20.0
1
2
3
4
5
6
1.025
1.000
0.975
0.950
0.925
0.900
0.875
0.850
Cfx
1: Mn = +0.12 S* = 0.12
2: Mn = +0.12 S* = 0.00
3: Mn = +0.00 S* = 0.12
4: Mn = +0.00 S* = 0.00
5: Mn = - 0.12 S* = 0.12
6: Mn = - 0.12 S* = 0.00
6
7
8
9
10
11
12
ξ

FIG 7 CASE II

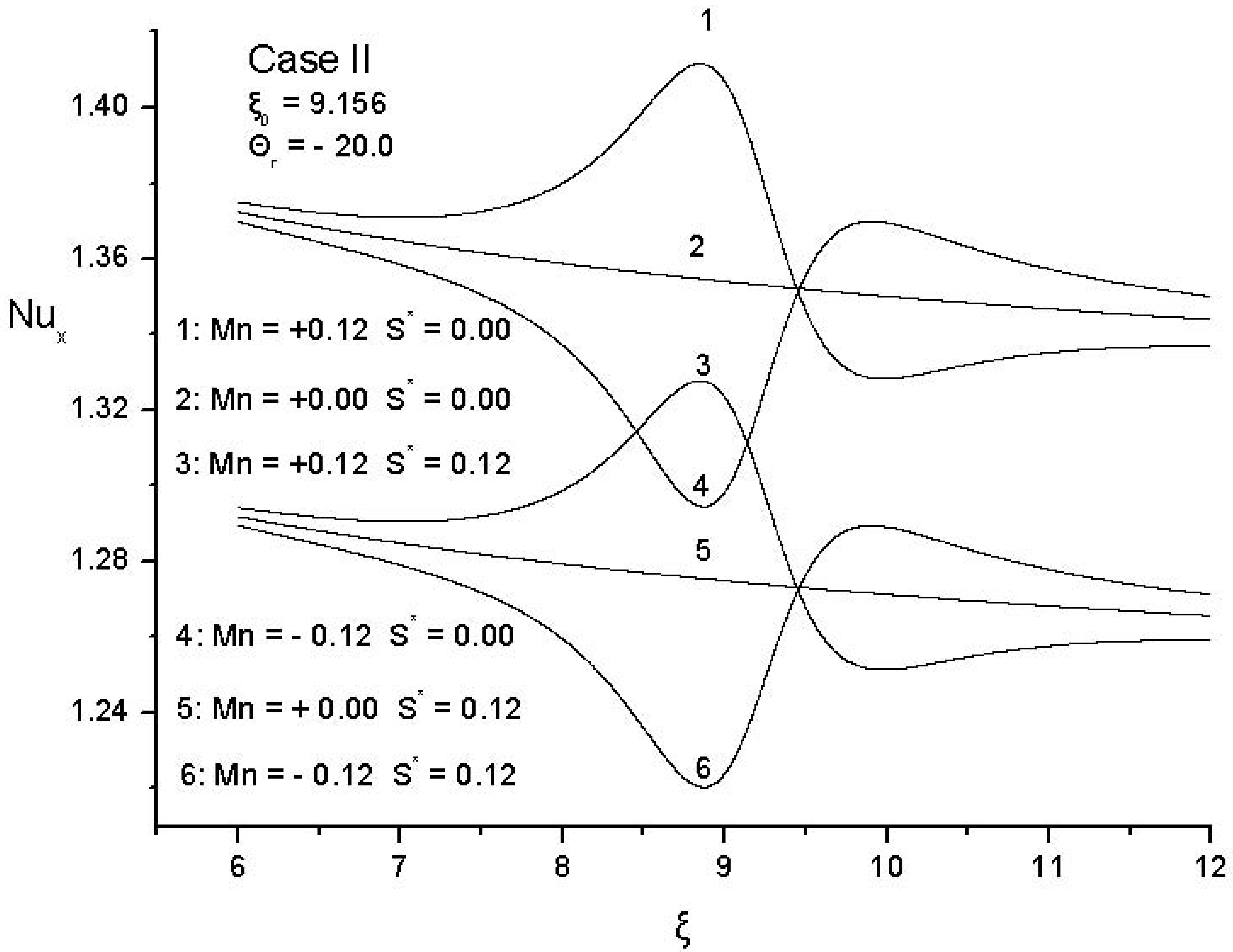

Case II
ξ0 = 9.156
Θr = - 20.0
1: Mn = +0.12 S* = 0.00
2: Mn = +0.00 S* = 0.00
3: Mn = +0.12 S* = 0.12
4: Mn = - 0.12 S* = 0.00
5: Mn = + 0.00 S* = 0.12
6: Mn = - 0.12 S* = 0.12
1
2
3
4
5
6
Nux
ξ
1.40
1.36
1.32
1.28
1.24
6
7
8
9
10
11
12

FIG 8 CASE II