PERGAMON

Chaos, Solitons and Fractals 14 (2002) 87–95

CHAOS
SOLITONS & FRACTALS

www.elsevier.com/locate/chaos

Interactions and stability of solitary waves in shallow water

E. Tzirtzilakis, M. Xenos, V. Marinakis, T.C. Bountis *

Department of Mathematics, Center for Research and Application of Nonlinear Systems, University of Patras, 26500 Patras, Hellas, Greece

Accepted 23 August 2001

Communicated by G. Nicolis

Abstract

In this work we use the spectral methods (s.m.) of numerical analysis to study solitary wave solutions of a nonlinear partial differential equation (pde), which is non-integrable and has been proposed as an improved approximation of shallow water wave propagation compared with the KdV equation. For sufficiently small parameters its solitary waves appear to be stable under time evolution and interact elastically as if they were pure solitons. This behaviour is probably due to the fact that this non-integrable pde can be transformed to an integrable equation with the aid of a nonlinear local transformation. As in the case of the KdV equation, when their speed increases, these wave solutions become unstable. However, unlike the KdV, the solitary waves of this new pde, in general, require a non-zero background which implies that they have infinite energy and thus may be unphysical. For any given values of the equation parameters these waves tend to zero exponentially at infinity, and thus represent a continuation of KdV solitons, only for one value of their velocity.

1. Introduction

The one-dimensional motion of solitary waves of inviscid and incompressible fluids has been the subject of research for more than a century [1]. The first observation of such waves was made by Russell [2] in 1834, who chased them on horseback along an Edinburgh channel. However, it was not till the 1870s that his work was finally recognized and studied further by Lord Rayleigh and Boussinesq [3–6]. Since that time many others worked on this discovery and probably one of the most important results was the derivation of the famous KdV equation by Korteweg and de Vries [7] in 1895. However, a wide class of solutions of this equation was difficult to obtain due to the nonlinearity of the pde. A very important step in this direction was made in 1965 with the numerical discovery of soliton solutions of the KdV by Kruskal and Zabusky [8]. Soon thereafter great progress was made towards the complete integration of this equation by the ingenious discovery of the Inverse Scattering Transform [9,10].

The KdV equation represents an approximation in the study of long wavelength, small amplitude inviscid and incompressible fluids. If one allows for the appearance of higher-order terms a more complicated equation is obtained, which is non-integrable but still admits some special wave solutions [11]. This equation, which will be referred to as generalized KdV, was studied in [12] by Fokas, who presented a local transformation connecting it with an integrable pde.

The development of spectral methods (s.m.) has considerably helped the study of the soliton equations and soliton solutions, with the combination of analytical and numerical approaches. Recent advances in computation have also made possible the development of s.m. and the more accurate numerical study of soliton equations. Classical engineering problems demand information about the spectral content of signals. For example, sonar signals of ships contain

* Corresponding author.
E-mail address: bountis@math.upatras.gr (T.C. Bountis).

0960-0779/02/$ - see front matter
PII: S0960-0779(01)00211-9

sinusoidal functions, due to the movement of the ship's propeller. The development of efficient algorithms therefore for fast implementation of discrete transformations has led to a great number of applications of spectral analysis to medicine, thermodiagrams, radars, acoustics, sonar, image analysis, construction vibration and systems design and analysis. These algorithms reduce computation time, accumulation errors and demand for memory and contribute to the development of digital hardware which has become widely available due to a decrease in the cost and size of semiconductors [13,14].

In this work we combine s.m. in space and a finite difference scheme in time to investigate numerically interactions of solitary wave solutions of the generalized KdV equation and compare them with the KdV equation. In the case of the KdV we first verify the elastic behaviour of solitary waves upon collision, using the numerical scheme. We then turn to the generalized KdV equation and find that solitary waves retain their fundamental features upon collision for small values of the parameters, representing its deviation from the KdV.

In Section 2 we give the mathematical formulation of the two problems under consideration and in Section 3 we describe the numerical methods which have been used. The numerical investigation is presented in Section 4. Our conclusions are given in Section 5, where we suggest that, owing to their behaviour at infinity, the solitary waves of the generalized water wave equation represent physically meaningful continuations of the KdV solitons only for one particular value of their velocity.

2. Mathematical formulation

Let us consider first the famous KdV equation in the following form

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \alpha u \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} = 0. \tag{1}$$

This equation represents a first approximation of unidirectional wave motion on the surface of a thin layer of an inviscid and incompressible fluid. The function $u(x,t)$ represents the amplitude of the fluid surface, while α and β characterize, respectively, the long wavelength and short amplitude of the waves.

It is well known that the above equation admits the exact solitary wave solution

$$u(x,t) = \frac{3(c-1)}{\alpha} \operatorname{sech}^2\left[\frac{1}{2}\sqrt{\frac{c-1}{\beta}}(x - ct - x_0)\right], \tag{2}$$

where c is the parameter that determines the propagation speed of the wave and x_0 is an arbitrary constant [6,15].

If we include second-order terms in α and β, Eq. (1) obtains the more physically realistic form [12]

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \alpha u \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} + \alpha^2 \rho_1 u^2 \frac{\partial u}{\partial x} + \alpha\beta\left(\rho_2 u \frac{\partial^3 u}{\partial x^3} + \rho_3 \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\right) = 0, \tag{3}$$

where ρ_1, ρ_2, ρ_3 are free parameters. In [12] it was observed that Eq. (3) can be transformed by the local change of coordinates

$$u = v - \alpha\rho_1 v^2 - \beta\left(3\rho_1 + \frac{7}{4}\rho_2 - \frac{1}{2}\rho_3\right)\frac{\partial^2 v}{\partial x^2} \tag{4}$$

to the completely integrable pde

$$\frac{\partial v}{\partial t} - \frac{3}{2}\beta\rho_2 \frac{\partial^3 v}{\partial x^2 \partial t} + \beta\left(1 - \frac{3}{2}\rho_2\right)\frac{\partial^3 v}{\partial x^3} + \alpha v \frac{\partial v}{\partial x} - \frac{1}{2}\alpha\beta\rho_2\left(v\frac{\partial^3 v}{\partial x^3} + 2\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}\right) = 0 \tag{5}$$

and the equivalence between the pdes (3) and (5) was established up to cubic terms in the small parameters α and β. Eq. (5) was first derived in [16], using the method of bi-hamiltonian systems and was later rederived from physical considerations in [17], where its Lax Pair was also given.

Unfortunately, as mentioned in [11,18], Eq. (3) is no longer integrable. However, it still admits the traveling wave solution

$$u(x,t) = K + \frac{3\beta k(A^2 - 2k)(2\rho_2 + \rho_3)\operatorname{sech}^2\left[\sqrt{k}(x - Ct - x_0)/\sqrt{2}\right]}{\alpha\rho_1\left(A - \sqrt{2k}\tanh\left[\sqrt{k}(x - Ct - x_0)/\sqrt{2}\right]\right)^2}, \tag{6}$$

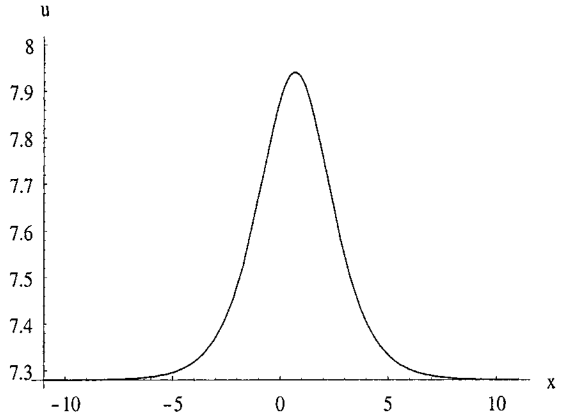

Fig. 1. The solitary wave profile of solution (6)–(8) of Eq. (3) at $t = 0$, for $\alpha = 0.6$, $\beta = 0.3$, $\rho_1 = \rho_2 = 1$, $\rho_3 = -0.9$, $x_0 = 0$, $A = 3$ and $k = 0.4$.

where

$$K = \frac{2\rho_1 - 2\rho_2 - \rho_3}{2\alpha\rho_1(\rho_2 + \rho_3)} - \frac{\beta(2\rho_2 + \rho_3)k}{\alpha\rho_1}, \tag{7}$$

$$C = \frac{4\rho_1 - 1}{4\rho_1} + \frac{(\rho_2 - 2\rho_1)^2}{4\rho_1(\rho_2 + \rho_3)^2} + \frac{\beta^2\rho_3(2\rho_2 + \rho_3)k^2}{\rho_1} \tag{8}$$

and A, k, x_0 are arbitrary constants [11].

Therefore, the initial condition that we use to solve numerically Eq. (3) can be extracted from the above relation for $t = 0$. We should also mention that, in order to have a bounded solitary wave solution, parameters A and k must satisfy $k > 0$ and $A > \sqrt{2k}$ or $A < -\sqrt{2k}$ (see Fig. 1). Moreover, the extremum of the solitary wave is maximum iff $-3\beta(2\rho_2 + \rho_3)k^2/(\alpha\rho_1) < 0$ [11].

Let us observe first that it does not appear a simple matter to let the $\rho_i \to 0$ in (6)–(8) and derive the KdV soliton (2) directly. The main reason for this is due to the fact that the additional terms in (3) (compared with (1)) endow the solitary wave solution (6) with a different functional form than (2) and do not allow their correspondence to be shown by a trivial limiting procedure. We have observed, however, numerically that the shapes of the two solutions (6) and (2) for small ρ_i are nearly identical.

It is also important to note that, as one can easily see from (6), these solitary waves possess a non-zero "background" given by (7) to which the amplitude tends exponentially as $|x| \to \infty$. This means that, in general, these waves will have infinite energy (taken over the full real line) and therefore be unphysical. Observe that if one wishes to remove this undesirable feature this can be done only for one particular value of k (or velocity of the wave) for any given set of parameters of the equation.

Before studying these solutions numerically, we will first give a brief presentation of the methods we use and discuss the results we obtain by comparing the solutions (2) and (6) of Eqs. (1) and (3), respectively.

3. The numerical method

S.m. have been developed from the main idea of the Galerkin method, which is a member of the class of weighted residuals methods (W.R.M.) [13]. These methods assumes that the solution can be represented analytically, in contrast with finite difference methods which define a solution only at a set of nodal points.

For our problem, we shall assume that the solution can be written as a sum

$$u(x,t) = \sum_{j=1}^{J} \alpha_j(t)\Phi_j(x), \tag{9}$$

where $\alpha_j(t)$ are unknown coefficients and $\Phi_j(x)$ are known analytic functions. The $\Phi_j(x)$ are often referred to as trial functions and (9) as trial solution. By forcing the analytic behaviour to be of the form (9) some error is introduced unless J is made arbitrarily large.

The starting point for a W.R.M. is to postulate an approximate solution. Eq. (9) can thus be extended by writing

$$u(x,t) = u_0(x,t) + \sum_{j=1}^{J} \alpha_j(t)\Phi_j(x), \tag{10}$$

where $u_0(x,t)$ is chosen to satisfy the boundary and initial conditions exactly, if possible. The approximating (trial) functions $\Phi_j(x)$ satisfy the orthogonality relation

$$\int_{-\infty}^{\infty} \Phi_j(x)\Phi_m(x)\,\mathrm{d}x \begin{cases} = 0 & \text{if } m \neq j \\ \neq 0 & \text{if } m = j \end{cases}$$

and the coefficients α_j are to be determined by solving a system of equations generated from the governing equation. For time dependent problems a system of ordinary differential equations is solved for $\alpha_j(t)$.

Let $\bar{u}$ be the exact solution of the KdV equation, satisfying

$$\bar{u}_t + \bar{u}_x + \alpha\bar{u}\bar{u}_x + \beta\bar{u}_{xxx} = 0. \tag{11}$$

If the approximate solution (10) is substituted into (11) it will not, of course, give zero. Thus we write

$$u_t + u_x + \alpha u u_x + \beta u_{xxx} = R,$$

where R is referred to as the equation residual. R is a continuous function of x and t. If J is made sufficiently large then, in principle, the coefficients $\alpha_j(t)$ can be chosen so that R can be made as small as we wish over the computational domain.

The coefficients $\alpha_j(t)$ are determined by requiring that the integral of the weighted residual over the computational domain is zero, i.e.

$$\int W_m(x) R\,\mathrm{d}x = 0, \quad m = 1,2,\ldots,J. \tag{12}$$

Thus a system of J algebraic equations for the α_j's is generated for the case considered here. According to the Galerkin Method the weight functions are chosen from the same family as the approximating (trial) functions, i.e.

$$W_m(x) = \Phi_m(x).$$

Relation (12) then indicates that the residual is orthogonal to every member of a complete set. Consequently as J tends to infinity the approximate solution u will converge to the exact solution $\bar{u}$. In order to construct trial functions we often use Fourier series or Legendre and Chebyshev polynomials.

The Galerkin spectral method produces very accurate solutions with relatively few unknown coefficients α_j in the approximate solution (10). However, when nonlinear terms are involved the evaluation of products of approximate solutions becomes very time-consuming. This lack of economy motivates the additional use of a collocation method within the Galerkin formulation. Collocation facilitates seeking the solution in terms of nodal unknown coefficients α_j in the approximate solution. The explicit use of nodal unknowns also permits boundary conditions to be incorporated more efficiently than the Galerkin spectral method. In the literature these collocation s.m. are usually referred to as Pseudospectral methods [19].

In order to examine the accuracy and efficiency of the pseudospectral method we will use in this paper, we apply it first to the KdV equation

$$u_t + u_x + \alpha u u_x + \beta u_{xxx} = 0 \tag{13}$$

with initial condition $u(x,0) = f(x)$. The time derivative in (13) is discretised using a finite difference approximation, in terms of central differences

$$u^{n+1} = u^{n-1} - 2\Delta t\left(u_x^n + \alpha u^n u_x^n + \beta u_{xxx}^n\right) = 0. \tag{14}$$

The pseudospectral method requires that (14) is satisfied at each collocation point x_j and introduces the approximate solution

$$u(x,t) = \sum_{k=0}^{N} \alpha_k(t)\Phi_k(x) \tag{15}$$

that will allow spatial derivatives to be evaluated. The trial functions used in (15) are $\Phi_k(x) = \mathrm{e}^{\mathrm{i}kx}$.

The steps used by the pseudospectral method to advance the solution from time level n to $n+1$ are:

(i) Given $u_j^n = u(x_j, t_n)$ evaluate $\alpha_k^n = \alpha_k(t_n)$ from (15).

(ii) Given α_k^n evaluate the derivatives e.g. $[\partial^2 u/\partial x^2]_j^n$ from (15).

(iii) Evaluate the nonlinear terms e.g. $u_j^n[\partial u/\partial x]_j^n$.

(iv) Evaluate u_j^{n+1} from (14), at $x = x_j$, $t = t_{n+1}$.

It can be seen that step (i) is the transformation from physical space to spectral space. Step (ii) occurs in spectral space and the evaluation of the nonlinear term in step (iii) is in physical space, which avoids the expensive multiplication of individual coefficients in the expansion of the form (15). Step (iv) occurs again in physical space.

The expansion in terms of an orthogonal system introduces a linear transformation between u and the sequence of its expansion coefficients $\{\alpha_k\}$. This is usually called the finite transform of u between physical space and transform space (spectral space). If the system is complete in a suitable Hilbert space, this transform can be inverted. Hence, functions can be described both through their values in physical space and through their coefficients in transform space [20].

For some of the most common orthogonal systems, such as Fourier and Chebyshev, the discrete transform can be computed in a "fast" way, i.e. with an operation count $(5/2)N\log_2 N$, where N is the number of polynomials, rather than with the $2N^2$ operations required by a matrix–vector multiplication [14].

In the current study we use the fast Fourier transform (FFT) described in [19] with a number of polynomials (points) $N = 1024$. The spatial step is chosen to be $\Delta x = 1$ and the time step $\Delta t = 0.001$.

4. The numerical investigation

In this section we investigate numerically the behaviour of the wave solutions for the two Eqs. (1) and (3) mentioned above, for several values of their parameters. As was mentioned in the previous section, we use central differences for the time variable t with accuracy $O((\Delta t)^2)$ and s.m. for the space variable x with accuracy $O(1/N)^N$ [21].

4.1. The KdV equation

We begin the investigation of KdV taking as initial condition the solitary wave (2) at $t = 0$ with $\alpha = 1$, $\beta = 0.1$, $x_0 = 20$ and $c = 1.1$. The stability condition for a finite difference scheme is described from the von Neumann analysis and is given by the relation [19]

$$\frac{\Delta t}{(\Delta x)^3} < \frac{1}{\pi^3} \approx 0.03225. \tag{16}$$

Since we have taken $\Delta x = 1$ and $\Delta t = 0.02$ condition (16) is satisfied. We observe that our wave moves along the spatial direction retaining its initial profile for a long time period, at least for $t = 2.5 \times 10^6$ time units (see Fig. 2).

In Fig. 3 a three-soliton interaction is shown for $c = 1.13$, $x_0 = 15$ (faster), $c = 1.08$, $x_0 = 35$ and $c = 1.03$, $x_0 = 57$ (slower). As was expected from the known soliton behavior of the KdV, the three waves interact elastically and remain unchanged before and after their interaction.

We also mention that similar results are obtained for various values of α, β and c, verifying that the code we use reproduces accurately the fundamental properties of the KdV.

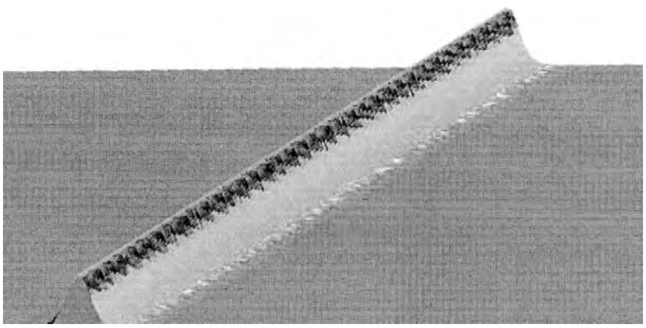

Fig. 2. The numerical integration of KdV with initial condition (2) at $t = 0$ with $\alpha = 1$, $\beta = 0.1$, $c = 1.1$ and $x_0 = 20$.

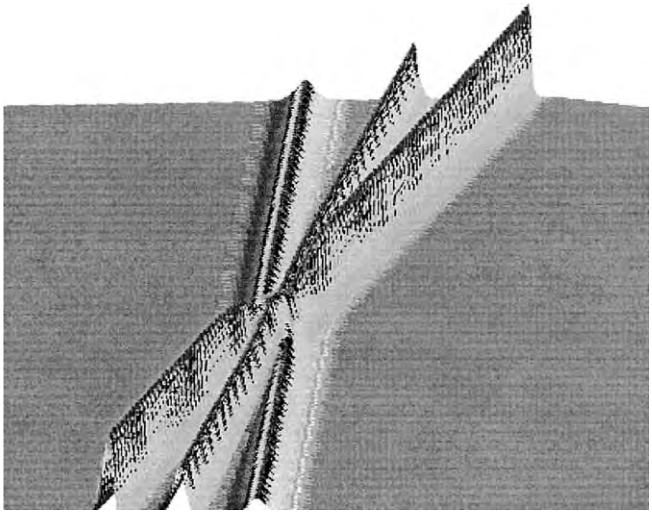

Fig. 3. Three wave interaction for the KdV.

4.2. The generalized KdV equation

Let us now proceed with the numerical study of the generalized KdV (3). In all that follows k satisfies the relations $k > 0$ and $A > \sqrt{2k}$ which are vital in order to have a bounded wave solution (6). Parameters A and α do not seem to affect the stability of the wave and for all cases we take $A = 3$ and $\alpha = 0.6$ (observe that α can be eliminated by re-scaling). Also in the numerical integration for one wave as initial position we take $x_0 = 30$.

First we set $\rho_1 = \rho_2 = 1$, $\rho_3 = -0.9$, $k = 0.8$ and observe that the wave remains stable up to $\beta \approx 0.44$ as unbounded oscillations appear beyond this value. Note that this wave instability is different from the numerically observed blow-up occuring when condition (16) is violated: Indeed true wave instability manifests itself in our computations through the appearance of increasing oscillations of "radiation waves", rather than the steady growth of the wave's maximum amplitude associated with numerical instability. Taking now the same values for ρ_i and $\beta = 0.4$ we observe a similar behaviour for the critical value $k \approx 0.87$. In Fig. 4 we thus present the stability region of these wave solutions for various values of β and k.

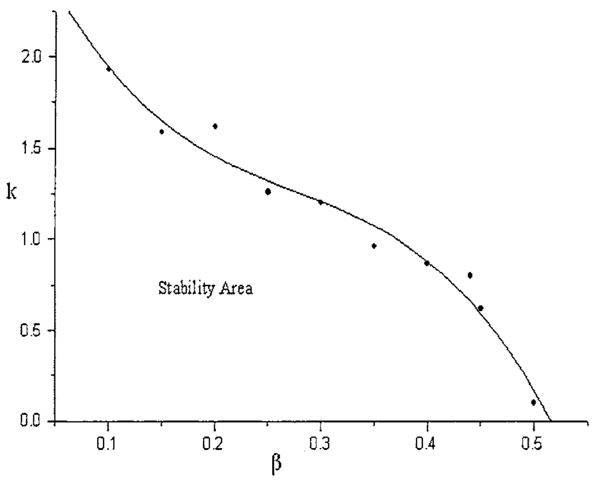

Fig. 4. The stability region of the solitary waves (6) with $\rho_1 = \rho_2 = 1$, $\rho_3 = -0.9$ and $\alpha = 0.6$ in β, k parameter plane.

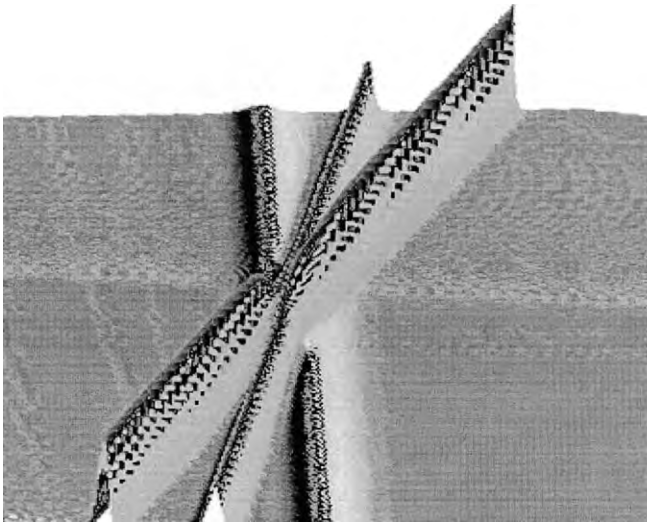

Fig. 5. Three wave interaction for the generalized KdV for $\alpha = 0.6$, $\beta = 0.4$, $\rho_1 = \rho_2 = 1$, $\rho_3 = -0.9$ and $k = 0.8$ (faster), $k = 0.4$, and $k = 0.065$ (slower).

On the other hand since the ρ_i are multiplied with α and β in (3) and also in the transformation (4) they cannot be arbitrarily large. Actually, for $\beta = 0.4$, $k = 0.4$ and $\rho_2 = \rho_1$, $\rho_3 = -0.9\rho_1$ we observe that the wave remains stable up to $\rho_1 \approx 7.3$.

We study now the interaction of two or three such wave solutions of Eq. (3). In contrast to the KdV, here the limits of these solutions when $|x| \to \infty$ depend not only on the free parameters of the equation, but also on k. This fact means that, when taking two or three waves, we are forced to make a slight displacement to avoid the discontinuities of the initial condition. However, this does not seem to affect the stability of the waves.

For relatively small values of β and k we observe that the waves retain their features not only when they move alone, but also when they interact with each other. For example in Fig. 5 we show the interaction of three waves for $\rho_1 = \rho_2 = 1$, $\rho_3 = -0.9$, $\alpha = 0.6$, $\beta = 0.4$ and $k = 0.8$, $x_0 = 10$ (faster), $k = 0.4$, $x_0 = 40$ and $k = 0.065$, $x_0 = 70$ (slower).

We also mention that we observe here again the appearance of oscillations as β or k increase, manifesting the occurence of wave instability. Moreover, all the above observations hold for various values of the time step within the interval $0.0001 < \Delta t < 0.005$.

4.3. Generalized KdV versus KdV

As was already mentioned the wave solution (6) has, in general, a non-zero background given by (7), in contrast with the solution (2) of the KdV. Of course, given some specific values for ρ_i and β, K becomes zero only for one specific value of k, suggesting that these waves represent true continuations of the KdV solitons only for a specific value of their velocity. These zero-background waves have a shape which is nearly identical to that of the KdV soliton and are also not always stable. For example, for $\alpha = 0.6$, $\beta = 0.4$, $\rho_1 = \rho_2 = 1$ and $\rho_3 = -0.9$ the corresponding k is 10.2273 which is far from the stability area shown in Fig. 4.

In Table 1 we show the stability or instability of the zero background wave (6) for $\alpha = 0.6$, $\beta = 0.4$ and various values of ρ_i. It can be seen that the stability of these zero-background waves depends on the corresponding k. In other words, whatever the values of ρ_i, β and k must be "sufficiently" small. Most probably for $\beta = 0.4$, if the corresponding k is less than 0.4, the wave is stable.

In order to examine interactions of three solitons with zero background we select $\alpha = 0.6$, $\beta = 0.4$, $\rho_1 = \rho_2 = 1$, $\rho_3 = -0.2$ and $k = 0.2$ (faster), $k = 0.173611$ (the corresponding k for zero background), $k = 0.05$ (slower). We not only observe a soliton-like behaviour, but we obtain a picture analogous to that shown in Fig. 3 rather than Fig. 5. The barely visible lines of numerical errors no longer exist even when taking $\Delta t = 0.01$, as it happens with the classical KdV.

Finally, we considered small values of the ρ_i parameters so that the generalized KdV approaches asymptotically the classical KdV. Taking $\rho_1 = \rho_2 = 0.01$ we found for $-0.000052 \leqslant \rho_3 < 0$ a stable solitary wave. For $\rho_3 = -0.000052$ the corresponding k for the zero-background wave is 0.32755. Calculating moreover the constant C from (8) and replacing it in (2) we construct an initial condition for the KdV which gives a stable solitary wave not only for the KdV, but also

Table 1
($\alpha = 0.6,\ \beta = 0.4$)

ρ_1	ρ_2	ρ_3	$\approx k$	Stable (+) unstable (−)
1.25	1.25	−0.5	0.416667	−
1.25	1.25	−0.4	0.280112	+
1.25	1.25	−0.3	0.179426	+
1	1	−0.4	0.520833	−
1	1	−0.35	0.407925	+
1	1	−0.2	0.173611	+
0.5	0.5	−0.2	1.041667	−
0.5	0.5	−0.1	0.347222	+
0.5	0.5	−0.05	0.146198	+
0.25	0.25	−0.1	2.083333	−
0.25	0.25	−0.05	0.694444	−
0.25	0.25	−0.01	0.106292	+

for the generalized Eq. (3). For both equations the time step is $\Delta t = 0.01$ and the numerical "behavior" is qualitatively the same.

5. Conclusion

In this work we have performed a comparative study of solitary wave solutions of the KdV and the generalized KdV equation. For the KdV the results are exactly as expected due to the well-known solitonic nature of the solutions.

For the generalized KdV we also observe a soliton-like behaviour for small β and k, which verifies the significance of the local transformation of the equation to a completely integrable one. Notice that, although this transformation is valid for small α and β, parameter α can be eliminated through a rescaling and does not affect the numerical study.

The most significant of the free parameters seems to be k, which appears only in the solutions and not in Eq. (3). When k is less than 0.4 we have observed that the wave solutions remain stable and retain their fundamental features after collision.

However, in general, solution (6) has a non-zero background, which means that it may be physically unrealistic. In order to examine waves with zero background, k can only take one value for specific α, β and ρ_i, so that $K = 0$, cf. (7). These waves "behave" numerically in exactly the same way with the solitons of the classical KdV. Moreover, when taking $\rho_i \to 0$, which means that the generalized Eq. (3) tends to KdV, we obtain a stable solitary wave for both KdV and generalized KdV again with the same dynamical properties.

We thus conclude that the physical importance of Eq. (3) must be further studied to assess whether its implications can be experimentally verified, or higher-order terms need to be introduced to make its predictions more physically plausible (see for example [12,22]).

Acknowledgements

We thank Mr. T. Skiniotis for his assistance with the use of pseudospectral techniques. We also acknowledge the support by PENED99-146 and 97EL16 programs of the GSRT of the Greek Ministry of Development and the Karatheodory research programs of the University of Patras under project No. 1938 and 2439.

References

[1] Whitham GB. Linear and nonlinear waves. New York: Wiley; 1974.
[2] Russel JS. The wave of translation in the oceans of water, air and ether. London: Trubner; 1895 (the 1844 Report on Waves).
[3] Rayleigh L. On waves. Philos Mag 1876;1(5):257–79.
[4] Boussinesq J. Essai sur la Theorie des eaux Courantes Mem. presentes par divers savants a l'Acad. Sci. (Paris), vol. 23;1877. p.1–680.
[5] Newell AC. Solitons in mathematics and physics. Philadelphia: SIAM; 1985.
[6] Drazin PG, Johnsom RS. Solitons: an introduction. Cambridge: Cambridge University Press; 1989.

[7] Korteweg D, de Vries G. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. Philos Mag 1895;39(5):422–43.
[8] Zabusky NJ, Kruskal MD. Interactions of 'solitons' in a collisionless plasma and the recurrence of initial states. Phys Rev Lett 1965;15:240–3.
[9] Gardner CS, Greene JM, Kruskal MD, Miura RM. Method for solving the Korteweg – de Vries equation. Phys Rev Lett 1967;19:1095–7.
[10] Ablowitz MJ, Segur H. Solitons and the inverse scattering transform. Philadelphia: SIAM; 1981.
[11] Marinakis V, Bountis TC. Special solutions of a new class of water wave equations. Comm Appl Anal 2000;4(3):433–45.
[12] Fokas AS. On a class of physically important integrable equations. Physica 1995;87D:145–50;
see also Fokas AS, Liu QM. PRL 1996;77(12):2347.
[13] Fletcher CAJ. Computational Techniques for Fluid Dynamics 1988;I.
[14] Elliot DF, Rao KR. Fast transforms, algorithms, analyses, applications. Orlando: Academic Press; 1982.
[15] Davis HT. Introduction to nonlinear differential and integral equations. New York: Dover; 1962.
[16] Fuchssteiner B, Fokas AS. Symplectic structures, their Bäcklund transformations and hereditary symmetries. Physica 1981;4D: 47–66.
[17] Camassa R, Holm DD. An integrable shallow water equation with peaked solitons. Phys Rev Lett 1993;71(11):1661–4.
[18] Marinakis V, Bountis TC. On the integrability of a new class of water wave equations. In: Duncan DB, Eilbeck JC, editors. Proceedings of the Conference on Nonlinear Coherent Structures in Physics and Biology, Heriot-Watt University, Edinburgh, July 10–14, 1995. Published on WWW: http://www.ma.hw.ac.uk/solitons/procs/.
[19] Fornberg B. A practical guide to pseudospectral methods. Cambridge: Cambridge University Press; 1996.
[20] Canuto C, Hussaini MY, Quarteroni A, Zang TA. Spectral methods in fluid dynamics. Berlin: Springer; 1988.
[21] Boyd J. Chebyshev and fourier spectral methods. 2nd ed. New York: Dover; 1999.
[22] Christov CI, Maugin GA, Velarde MG. Well-posed Boussinesq paradigm with purely spatial higher-order derivatives. Phys Rev E 1996;54(4):3621–38.