Received January 2002
Revised April 2003
Accepted April 2003

# Numerical study of biomagnetic fluid flow over a stretching sheet with heat transfer

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***Keywords*** *Fluid dynamics, Magnetic fields, Finite difference methods*

***Abstract*** *In the present work, a Chebyshev pseudospectral method for the numerical solution of the two-dimensional, laminar, incompressible, viscous flow of a biomagnetic fluid over a stretching sheet under the action of an applied magnetic field and in the presence of heat transfer is applied and demonstrated. In this problem, it is assumed that the magneto-thermo-mechanical coupling is described not only by a function of temperature but by an expression involving also the magnetic field intensity. The numerical method used for the solution of the coupled and non-linear boundary value problem of ordinary differential equations, describing this physical problem, achieves high accuracy using relatively few nodal points. A comparison with numerical results, obtained by using a finite difference method is also made, showing the efficiency of the Chebyshev pseudospectral method.*

## Introduction

During the last decades, an extensive research work has been done on the fluid dynamics of biological fluids in the presence of magnetic field due to bioengineering and medical applications (Haik *et al.*, 1999; Plavins and Lauva, 1993; Ruuge and Rusetski, 1993).

Many contemporary problems of interest in this area as well as in other areas of fluid mechanics, are reduced, by the introduction of suitable similarity or pseudo-similarity variables, to a non-linear and coupled system of ordinary differential equations with their appropriate boundary conditions. A variety of numerical methods have been devised for dealing with such two-point boundary value problems.

The Runge-Kutta integration scheme, along with the Newton-Raphson shooting method, is one of the most commonly used techniques for the solution of such two-point boundary value problems (Gladwell and Sayers, 1980; Rahman, 1978; Shang and Wang, 1990; Soong and Hwang, 1990; Wang, 1990a, b). Even though this method provides satisfactory results for such types of

The authors wish to thank Professor N.G. Kafoussias, Department of Mathematics, University of Patras, for his helpful suggestions and constructive criticism. This work was partially supported from Karatheodoris research program of the University of Patras under Project No. 2439.

International Journal of Numerical Methods for Heat & Fluid Flow
Vol. 13 No. 7, 2003
pp. 830-848

0961-5539
DOI 10.1108/09615530310502055

problems, it may fail when applied to problems in which the differential equations are very sensitive to the choice of the missing initial conditions. Moreover, another serious difficulty which may be encountered in the boundary-value problems is the inherent instability. Difficulty also arises in the case in which one end of the range of integration is at infinity. The end-point of integration is usually approximated by assigning a finite value to this point; it is obtained by estimating a value at which the solution will reach its asymptotic state. The computing time for integrating differential equations sometimes depend critically on the quality of the initial guesses of the unknown boundary conditions and the infinite end-point.

Another numerical method which is used for the solution of such type of problems is based on the common finite differences (FDs) method. The essential features of this technique are the following:

(1) it is based on the common FDs method with central differencing,

(2) on a tridiagonal matrix manipulation, and

(3) on an iterative procedure (Kafoussias and Williams, 1993).

This method is simple and has better stability characteristics.

On the contrary to the above-mentioned numerical methods, the relatively new numerical technique which is used in the present study based on a Chebyshev pseudospectral method (PS method or CPSM), is accurate and efficient using few nodal points. Thus, CPSM is memory minimizing and faster, especially for multi-dimensional problems, and achieves higher accuracy than a FD-based method.

So, in the present study, the PS method for the numerical solution of the fundamental problem of the flow of a biomagnetic fluid over a stretching sheet, in the presence of an applied magnetic field due to a magnetic dipole, as in Andersson and Valnes (1998), is applied and demonstrated.

Moreover, it is assumed that the magneto-thermo-mechanical coupling is not only described by a function of temperature ($M = K^{*}(T_c - T)$), as in Andersson and Valnes (1998), but by an expression involving also the magnetic field strength $H$, used by Matsuki *et al.* (1977), ($M = KH(T_c - T)$). This equation, permit us not to consider the biofluid far away from the sheet at Curie temperature $T_c$ in order to have no further magnetization. This feature is essential for physical applications because the Curie temperature is very high, e.g. 1,043 K for iron, and such a temperature would be meaningless for applications concerning most of biofluids like blood. The mathematical formulation of the problem is obtained by an analogous manner presented in Andersson and Valnes (1998).

The numerical solution is obtained by using a simple and accurate CPSM-based method (Boyd, 2000) and is compared with another numerical technique based on the common FDs (Kafoussias and Williams, 1993). The obtained results are presented graphically, for different values of the parameters entering into the problem under consideration, and showed that the flow is appreciably influenced by the magnetic field. Moreover, with the use of the CPSM, we are able to obtain much more accurate results, with much less computational time and memory cost, than by using the FDs method.

## Mathematical formulation

We consider a viscous and electrically non-conducting biomagnetic fluid confined to the half space ($y > 0$) above a sheet. A magnetic dipole is located below the sheet giving rise to a magnetic field of sufficient strength to saturate the biomagnetic fluid. The sheet is flat, impermeable, elastic, stretched with a velocity proportional to the distance $x$ ($u = cx$) and kept at fixed temperature $T_w$. The fluid far away from the sheet is at rest and at temperature $T_c$, greater than $T_w$.

We assume laminar, steady, two-dimensional and incompressible biomagnetic fluid flow shown schematically in Figure 1. The equations are similar to those derived in ferrohydrodynamics (Rosensweig, 1985) and are the mass conservation, fluid momentum at $x$ and $y$ direction and energy equation:

$$\vec{\nabla} \cdot \vec{q} = 0, \tag{1}$$

$$\rho \vec{q} \cdot (\vec{\nabla} u) = -\frac{\partial p}{\partial x} + \mu_0 M \frac{\partial H}{\partial x} + \mu \nabla^2 u, \tag{2}$$

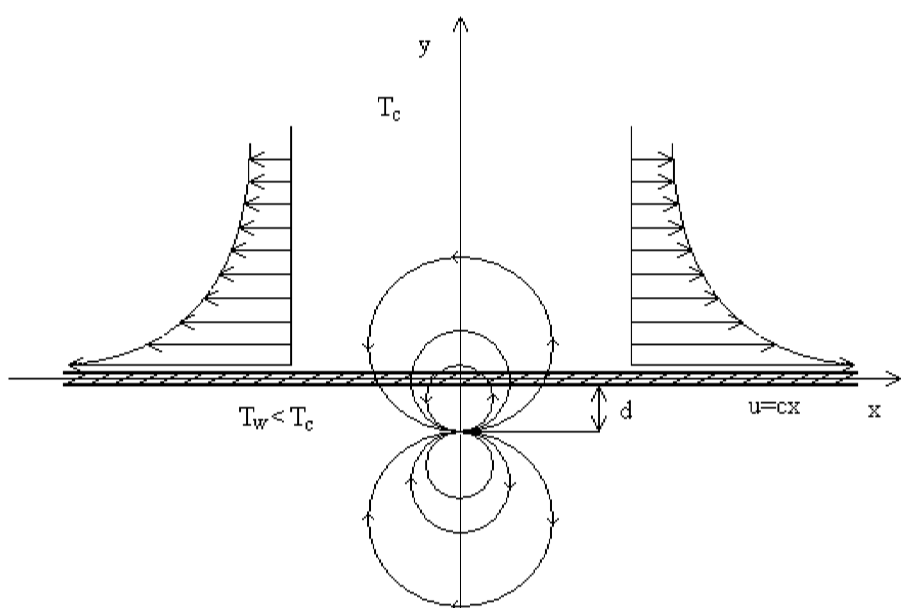


**Figure 1.** Schematic representation of flow configuration

$$\rho\vec{q}\cdot(\vec{\nabla}v) = -\frac{\partial p}{\partial y} + \mu_0 M\frac{\partial H}{\partial y} + \mu\nabla^2 v, \tag{3}$$

$$\rho c_p\,\vec{q}\cdot(\vec{\nabla}T) + \mu_0 T\frac{\partial M}{\partial T}\,\vec{q}\cdot(\vec{\nabla}H) = k\nabla^2 T + \mu\Phi, \tag{4}$$

with boundary conditions:

$$y = 0 \;:\; u = cx,\;\; v = 0,\;\; T = T_w \tag{5}$$

$$y \to \infty \;:\; u = 0,\;\; T = T_c,\;\; p + 1.2q^2 = \text{const.} \tag{6}$$

In the above equations, $u$ and $v$ are the velocity components of the fluid in the $x$ and $y$ direction, respectively ($\vec{q} = (u, v)$), $p$ the pressure, $\rho$ the biomagnetic fluid density, $\mu$ the dynamic viscosity, $\mu_0$ the magnetic permeability, $c_p$ the specific heat at constant pressure and $k$ the thermal conductivity. Also $\nabla^2$ is the two-dimensional Laplacian operator ($\nabla^2 = \vec{\nabla}\cdot\vec{\nabla} = \partial^2.\,\partial x^2 + \partial^2.\,\partial y^2$) and $\Phi$ is the dissipation function which in our case is given by

$$\Phi = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 \tag{7}$$

The terms $\mu_0 M(\partial H.\,\partial x)$ and $\mu_0 M(\partial H.\,\partial y)$ in equations (2) and (3), respectively, represent the components of the magnetic force per unit volume and depend on the existence of the magnetic gradient. When the magnetic gradient is absent these forces vanish. The second term, on the left-hand side of the thermal energy equation (4), accounts for heating due to adiabatic magnetization.

We consider that the magnetic dipole is located at distance $d$ below the sheet and we define the dimensionless distance $\alpha$ as $\alpha = (c\rho.\,\mu)^{1.2}d$. The magnetic dipole gives rise to a magnetic field, sufficiently strong to saturate the biofluid, and its scalar potential is given by Andersson and Valnes (1998)

$$V(x, y) = \frac{\alpha}{2\pi}\,\frac{x}{x^2 + (y + d)^2} \tag{8}$$

Thus, the magnitude $\|\vec{H}\| = H$ of the magnetic field intensity is given by

$$H(x, y) = \left[H_x^2 + H_y^2\right]^{1.2} = \frac{\gamma}{2\pi}\frac{1}{x^2 + (y + d)^2} \tag{9}$$

where $\gamma = \alpha$ and $H_x$, $H_y$ are the components of the magnetic field $\vec{H} = (H_x, H_y)$, given by

$$H_x(x,y) = -\frac{\partial V}{\partial x} = \frac{\gamma}{2\pi}\frac{x^2-(y+d)^2}{\left[x^2+(y+d)^2\right]^2}, \tag{10}$$

$$H_y(x,y) = -\frac{\partial V}{\partial y} = \frac{\gamma}{2\pi}\frac{2x(y+d)}{\left[x^2+(y+d)^2\right]^2}. \tag{11}$$

The gradients can be obtained from equation (9) after having expanded in powers of $x$ and retained terms up to $x^2$, thus

$$\frac{\partial H}{\partial x} = -\frac{\gamma}{2\pi}\frac{2x}{(y+d)^4}, \quad \frac{\partial H}{\partial y} = \frac{\gamma}{2\pi}\left[-\frac{2}{(y+d)^3}+\frac{4x^2}{(y+d)^5}\right]. \tag{12}$$

The magnetic field intensity $H$, can be expressed by analogous manner, as

$$H(x,y) = \frac{\gamma}{2\pi}\left[\frac{1}{(y+d)^2}-\frac{x^2}{(y+d)^4}\right]. \tag{13}$$

Under the assumption that the applied magnetic field $\vec{H}$ is sufficiently strong to saturate the biomagnetic fluid, the magnetization $M$ is, generally, determined by the fluid temperature and magnetic field intensity $H$. Andersson and Valnes (1998) considered that the variation of magnetization $M$ can be approximated, as a function of temperature $T$ by the linear equation of state $M = K^*(T_c - T)$, where $K^*$ is a constant called pyromagnetic coefficient and $T_c$ is the Curie temperature. When the biofluid reaches the Curie temperature, it is no longer subjected to further magnetization (magnetized), as the magnetic field intensity increases.

However, Matsuki *et al.* (1977) considered that the variation of magnetization $M$ is a function of temperature $T$ and magnetic field $H$ and proved, experimentally, that

$$M = KH(T_c - T). \tag{14}$$

It is worth remarking once more that equation (14), used in the current study, permit us not to consider the biofluid far away from the sheet at Curie temperature, $T_c$, in order to have no further magnetization. The Curie temperature is very high, e.g. 1,043 K for iron, and such a temperature would be meaningless for applications concerning most of the biofluids like blood. So, instead of having zero magnetization far away from the sheet, due to the increase of fluid temperature up to the Curie temperature, formulation (14) allows us to consider whatever temperature is desired and the magnetization will be zero due to the absence of the magnetic field sufficiently far away from the sheet.

*Transformation of equations*

We introduce the non-dimensional variables (Andersson and Valnes, 1998)

$$\Psi(\xi,\eta) = \left(\frac{\mu}{\rho}\right)\xi f(\eta), \quad p(\xi,\eta) = \frac{P}{c\mu} = -P_1(\eta) - \xi^2 P_2(\eta), \tag{15}$$

$$\Theta(\xi,\eta) = \frac{T_c - T}{T_c - T_w} = \Theta_1(\eta) + \xi^2\Theta_2(\eta), \tag{16}$$

where $\eta$, $\xi$ are the dimensionless coordinates defined as

$$\xi(x) = \left(c\frac{\rho}{\mu}\right)^{1.2} x, \quad \eta(y) = \left(c\frac{\rho}{\mu}\right)^{1.2} y \tag{17}$$

and $\Psi(\xi,\eta)$, $\Theta(\xi,\eta)$ and $P(\xi,\eta)$ are the dimensionless stream function, temperature and pressure, respectively.

The velocity components can be calculated as

$$u = \frac{\partial\Psi}{\partial y} = cxf'(\eta) \quad v = -\frac{\partial\Psi}{\partial x} = -\left(\frac{c\mu}{\rho}\right)^{1.2} f(\eta) \tag{18}$$

where $()' = \partial().\partial\eta$.

Substituting equations (12)-(18) into the momentum equations (2) and (3) and the energy equation (4) and equating coefficients of equal powers of $\xi$, up to $\xi^2$, we obtain the following system of ordinary differential equations

$$f''' + ff'' - (f')^2 + 2P_2 - \frac{2\alpha^2\Theta_1}{(\eta+\alpha)^6} = 0, \tag{19}$$

$$P_1' - f'' - ff' - \frac{2\alpha^2\Theta_1}{(\eta+\alpha)^5} = 0, \tag{20}$$

$$P_2' - \frac{2\alpha^2\Theta_2}{(\eta+\alpha)^5} + \frac{6\alpha^2\Theta_1}{(\eta+\alpha)^7} = 0, \tag{21}$$

$$\Theta_1'' + \Pr f\Theta_1' + \frac{\lambda 2\alpha^2(\Theta_1 - T_\varepsilon)f}{(\eta+\alpha)^5} + 2\Theta_2 - 4\lambda(f')^2 = 0, \tag{22}$$

$$\Theta_2'' - \Pr(2f'\Theta_2 - f\Theta_2') + \frac{\lambda 2\alpha^2 f\Theta_2}{(\eta+\alpha)^5}$$

$$-\lambda 2\alpha^2(\Theta_1 - T_\varepsilon)\left[\frac{f'}{(\eta+\alpha)^6} + \frac{f}{(\eta+\alpha)^7}\right] = 0, \tag{23}$$

Also, the boundary conditions (5) and (6) becomes

$$\eta = 0 \quad : \; f = 0, \; f' = 1, \; \Theta_1 = 1, \; \Theta_2 = 0, \tag{24}$$

$$\eta \to \infty \quad : \; f' \to 0, \; \Theta_1 \to 0, \; \Theta_2 \to 0, \; P_1 \to -P_\infty, \; P_2 \to 0. \tag{25}$$

The five dimensionless parameters appearing in the transformed equations are

$$\left.\begin{array}{ll}
\Pr = \mu c_{\mathrm{p}}.\, k & \text{(Prandtl number)}, \\
\lambda = \dfrac{c\mu^2}{\rho k(T_{\mathrm{c}} - T_{\mathrm{w}})} & \text{(viscous dissipation parameter)}, \\
T_\varepsilon = T_{\mathrm{c}}.\,(T_{\mathrm{c}} - T_{\mathrm{w}}) & \text{(dimensionless temperature)}, \\
\beta = \frac{\gamma}{2\pi}\mu_0 \frac{KH(0,0)\,(T_{\mathrm{c}} - T_{\mathrm{w}})\rho}{\mu^2} & \text{(biomagnetic interaction parameter} \\
\alpha = (c\rho.\,\mu)^{1.\,2} d & \text{(dimensionless distance } \alpha).
\end{array}\right\} \tag{26}$$

The system of equations (19)-(23), subjected to the boundary conditions (24) and (25), is a five-parameter coupled and non-linear system of ordinary differential equations, describing the biomagnetic fluid flow over a stretching sheet when the magnetization of the fluid is given as a function of temperature $T$ and magnetic field intensity $H$.

## Numerical method

In order to demonstrate the application of CPSM for the solution of the system of equations (19) and (23), subjected to the boundary conditions (24) and (25), we apply it to the first equation of the above-mentioned system. Equation (19) can be written as:

$$(f')'' + f(f')' - f'(f') = \frac{2\alpha^2\beta\Theta_1}{(\eta+\alpha)^6} - 2P_2 \tag{27}$$

which is of the form

$$P(x)u''(x) + Q(x)u'(x) + R(x)u(x) = Rhs(x) \tag{28}$$

or

$$\underbrace{\left[P(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + Q(x)\frac{\mathrm{d}}{\mathrm{d}x} + R(x)\right]}_{\mathbf{L}\quad u(x)=Rhs(x)} u(x) = Rhs(x) \tag{29}$$

where $\mathbf{L}$ is a linear differential operator, $u(x) = f'(\eta)$, $P(x) = 1$, $Q(x) = f(\eta)$, $R(x) = -f'(\eta)$ and

$$Rhs(x) = \frac{2\alpha^2\beta\Theta_1}{(\eta+\alpha)^6} - 2P_2.$$

In an analogous manner all equations of the system can be reduced in the form of equation (28) except for equations (20) and (21) which are of first order. The solution of equation (28) is obtained on the basis recombination method referred in Boyd (2000).

According to this method we assume that the solution of second order ODE of the form of equation (28), with the non-homogeneous boundary conditions,

$$u(-1) = \alpha, \quad u(1) = \beta, \tag{30}$$

can be expressed, in terms of $N+1$ functions $[\phi_n(x)]$, where $[\phi_n(x)]$ is, in general, the basis of any family of orthogonal polynomials. In our case, it is the family of the Chebyshev polynomials.

So, we assume that the solution of equation (28) can be expressed as

$$u(x_i) = \sum_{j=1}^{N} \alpha_j T_{j-1}(x_i) \quad i = 1, 2, \ldots, (N-2) \tag{31}$$

where $\alpha_j$ are the unknown coefficients to be determined, $T_{j-1}(x_i)$ are the Chebyshev polynomials and $x_i$ are the $N$-point Gauss-Lobatto grid points defined by

$$x_i = \cos\left(\frac{\pi i}{N-1}\right) \quad i = 1, 2, \ldots, (N-2)$$

The next step is to construct a function $B(x)$, which is arbitrary, but satisfies the above-mentioned boundary conditions (30), to define a new variable $v(x)$ and a new forcing function $g(x)$ such as

$$u(x) \equiv v(x) + B(x) \tag{32}$$

$$g(x) \equiv Rhs(x) - \mathbf{L}B(x) \tag{33}$$

so that the modified problem becomes

$$\mathbf{L}v = g \tag{34}$$

The new variable $v(x)$ satisfies the homogeneous boundary conditions

$$v(-1) = v(1) = 0$$

The *basis recombination* technique works by choosing simple linear combinations of the original basis functions so that these combinations, the new basis functions, individually satisfy the homogeneous boundary conditions.

A choice of basis functions, such that $\phi_n(\pm 1) = 0$, for all $n$, can be

$$\phi_{2n}(x) \equiv T_{2n}(x) - 1 \qquad n = 1, 2, \ldots$$
$$\phi_{2n+1}(x) \equiv T_{2n+1}(x) - x \quad n = 1, 2, \ldots$$

The grid points $x_i$ are defined in the same way as mentioned earlier, and the new function $v(x)$ can be written as

$$v(x) \approx \sum_{n=2}^{N-1} b_n \phi_n(x) \tag{35}$$

So, we define a column vector $\vec{b}$ of dimension $(N - 2)$ and the differential equation (28) is finally transformed to the matrix problem

$$\mathbf{A}\vec{b} = \vec{C} \tag{36}$$

where

$$A_{i,j} = P(x_i)\phi_{j+1,xx}(x_i) + Q(x_i)\phi_{j+1,x}(x_i) + R(x_i) \quad i, j = 1, 2, \ldots, (N - 2)$$
$$C_i = g(x_i) \qquad i = 1, 2, \ldots, (N - 2)$$

By solving the matrix problem (36) we obtain the unknown coefficients $b_n$ for $v(x)$ and by noting

$$a_n = b_n, \quad n \geq 2$$

$$a_0 = -\sum_{n=1}^{(2n)\leq(N-1)} b_{2n}, \quad a_1 = -\sum_{n=1}^{(2n+1)\leq(N-1)} b_{2n+1}$$

we are able to express the solution $v(x)$ as an ordinary Chebyshev series:

$$v(x) \equiv \sum_{n=0}^{N-1} a_n T_n(x)$$

Consequently, the solution $u(x)$ of the inhomogeneous problems (28)-(30) can be derived, as equation (32) denotes, by adding the function $B(x)$ to the obtained solution $v(x)$ of the homogeneous problem.

The solution, by using the Chebyshev polynomials, is obtained at the interval $[-1, 1]$ but it is a trivial problem to derive it at any interval $[a, b]$, where $a, b \in \mathbb{R}$, by using a linear transformation. The first order equations (20) and (21), as well as the derivation of $f$ from $f'$, calculated from equation (19), can be treated in a analogous manner as mentioned earlier. The basis functions are chosen such that $\Phi(-1) = 0$ or $\Phi(+1) = 0$, depending on which boundary is the known value for $u(x)$.

Having demonstrated the application of CPSM for the solution of a boundary value problem, for a second order ODE, we proceed for the application of this method for our coupled and non-linear boundary value problem described by equations (19)-(25).

The solution of the system (19)-(23), subjected to the boundary conditions (24) and (25), is obtained using the above-mentioned method in joint with a relaxation technique described in detail in Kafoussias and Williams (1993).

Hence, to start the solution procedure, it is necessary to construct distribution curves for $f^{/}$, $\Theta_1$, $\Theta_2$ between $\eta = 0$ and $\eta = \eta_\infty$ ($\eta \to \infty$) which satisfy the boundary conditions (24) and (25). These selected curves can be quite arbitrary as long as they satisfy the boundary conditions. Considering $\Theta_1$ and $\Theta_2$ as known, we obtain a distribution curve for $P_2$ from equation (21), which is of first order (integration). The $f$ distribution can be obtained also from $f^{/}$ curve with integration. The next step is to consider the $f$, $P_2$ and $\Theta_1$ known and to derive a new estimation for $f^{/}$ ($f^{/}_{\text{new}}$), by solving the non-linear equation (19) using the method described earlier. The distribution $f$ is updated by the integration of the new $f^{/}$ curve. These new profiles of $f^{/}$ and $f$ are then used for new inputs. So, equation (19) is solved iteratively until convergence, up to a small $\varepsilon$ is attained.

Considering now $f^{/}$ (or $f$) known equation (22) is solved by using the same algorithm, but without iterations now as far as equation (22) is linear considering $f$, and $\Theta_2$ known. So, we obtain a new approximation $\Theta_{1\text{new}}$ for $\Theta_1$ and in the same way a new approximation for $\Theta_2$. Also, considering $\Theta_1$ and $\Theta_2$ known, we obtain a new estimation for $P_2$.

Finally, the computational procedure reverts to its original starting point using the most current distributions of $f^{/}$, $\Theta_1$, $\Theta_2$ and $P_2$ as inputs. This process is continued until final convergence is attained and is represented schematically (flow-chart) in Figure 2. Equation (20) is not coupled with the other equations of the system (19)-(23) and so we exclude it from the iterative procedure.

## Results and discussion

In order to solve the coupled and non-linear system of equations (19)-(23) subjected to the boundary conditions (24) and (25), by using the pseudospectral algorithm described earlier, as well as a FD-based method described in detail in Kafoussias and Williams (1993), it is necessary to give the numerical values for the dimensionless parameters $\alpha$, $\beta$, $\lambda$ and $T_\varepsilon$ defined in equation (26). The values of the dimensionless parameters (26), entering into the problem under the consideration are $\alpha = 1$, $\beta = 0.0 - 5.0$, $\lambda = 0.001$ and $T_\varepsilon = 2$. For these

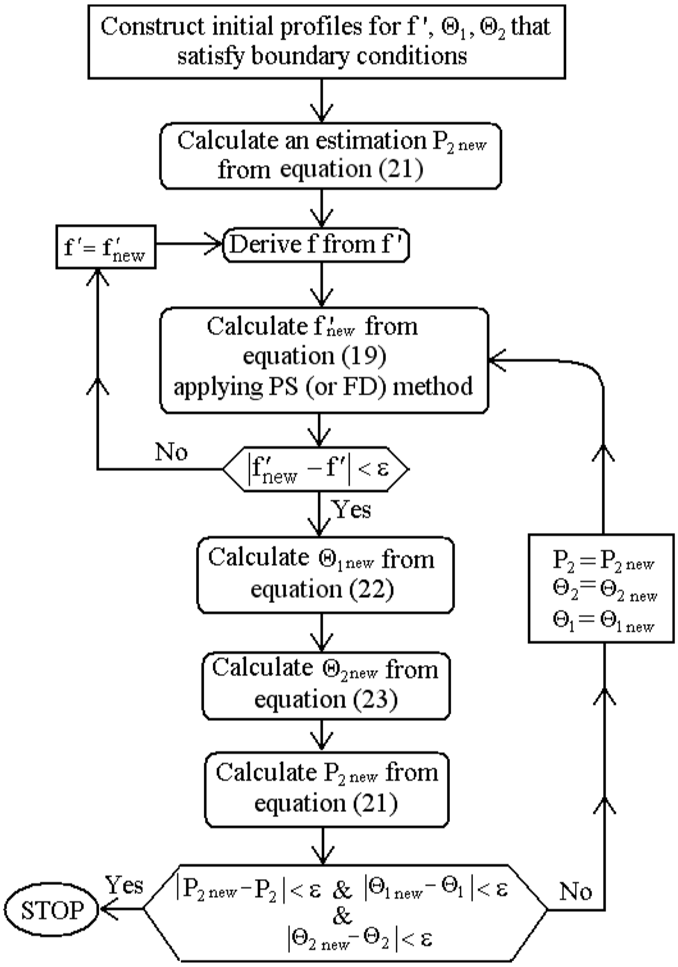


**Figure 2.**
Flow chart of the computer program

values of the dimensionless parameters $\alpha$, $\lambda$ and $T_\varepsilon$, variation of $\beta$ means variation of the intensity of the applied magnetic field. Although the viscosity $\mu$, the specific heat under constant pressure $c_p$ and the thermal conductivity $k$ of any fluid, and hence of the blood as a characteristic biofluid, are temperature dependent, Prandtl number $Pr = \mu c_p . k$ can be considered constant. Thus, for human body temperature $T = 310\,K$, the values of $\mu$, $c_p$ and $k$ are equal to $3.2 \times 10^3$ kg. m s, 14.65 J. kg K and $2.2 \times 10^{-3}$ J. m s K, respectively (Chato, 1980; Valvano *et al.*, 1994) and hence $Pr = 21$.

The most important flow and heat transfer characteristics are the local skin friction coefficient and the local rate of heat transfer coefficient. These quantities can be defined by the following relations:

$$C_{f_x} = -\frac{2\tau_w}{\rho(cx)^2} \text{ and } \mathrm{Nu}_x = \frac{x}{T_c - T_w}\frac{\partial T}{\partial y}\Big|_{y=0}, \tag{37}$$

where

$$\tau_w = \mu \frac{\partial u}{\partial y}\Big|_{y=0}$$

is the wall shear stress, $\mathrm{Nu}_x$ is the local Nusselt number and $\mathrm{Re}_x$ is the local Reynolds number defined as $\mathrm{Re}_x = \rho c x^2 . \mu$. Using equations (16)-(18), the above-mentioned quantities can be written as

$$C_{f_x} = -2f^{//}(0)\,\mathrm{Re}_x^{-1.2} \quad \text{and} \quad \mathrm{Nu}_x = -[\Theta_1^{/}(0) + \xi^2\Theta_2^{/}(0)]\,\mathrm{Re}_x^{1.2}, \tag{38}$$

where $f^{//}(0)$ is the dimensionless wall shear parameter and $\Theta^{/}(0) = (\Theta_1^{/}(0) + \xi^2\Theta_2^{/}(0))$ is the dimensionless *wall heat transfer parameter*. It is apparent that the flow field is affected by the biomagnetic interaction parameter $\beta$. In hydrodynamic case ($\beta = 0$), $P_2$ becomes a constant equal to its value zero at infinity (equations (21) and (25)) and on the other hand, the flow problem is decoupled from the thermal energy problem. So, it is more interesting and convenient to replace the dimensionless wall heat transfer parameter $-\Theta^{/}(0) = -[\Theta_1^{/}(0) + \xi^2\Theta_2^{/}(0)]$ by the dimensionless, and independent of the distance $\xi$, ratio

$$\Theta^*(0) = \frac{\Theta_1^{/}(0)}{\Theta_1^{/}(0)|_{B=0}},$$

called the coefficient of the *heat transfer rate at the wall* (*sheet*). Also, the quantity $P_2(0)$ can be defined as the *wall pressure parameter*.

The variations of the above-mentioned coefficients are obtained by using FD with step size $\Delta\eta = 0.01$. An appropriate $\eta_\infty$ value, as an approximation to $\eta = \infty$, is 6, and is used also for the computations with the PS method. These results are shown schematically in Figures 3-5 for the wall shear parameter, wall pressure parameter and wall heat transfer parameter, respectively, and obtained for $N = 22$ for the CPSM. It is observed from these figures that the above-mentioned coefficients varying almost linearly with the biomagnetic interaction parameter $\beta$ and $-f^{//}(0)$ as well as $P_2(0)$ increase with the increase of $\beta$ whereas $\Theta^*(0)$ decreases.

In Figure 6, the variation of $P_2$ with dimensionless distance $\eta$ is shown for various values of the biomagnetic parameter $\beta$. It can be observed that near the boundary, $P_2$ decreases rapidly as $\eta$ increases. This variation can be sufficiently expressed by the solution obtained using the CPSM despite the existence of few number of grid points (just five) nearer to this boundary as shown in Figure 7.

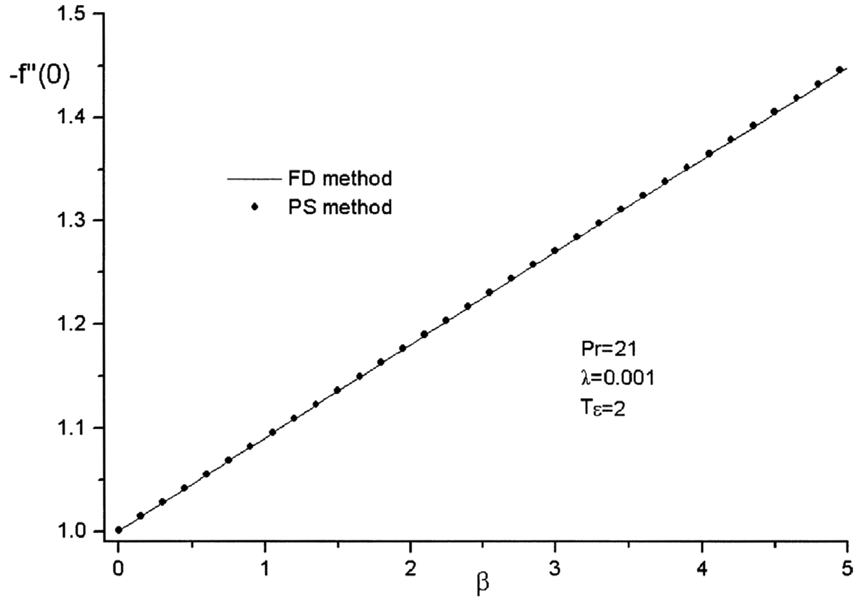


**Figure 3.**
Variation of the wall shear parameter $-f''(0)$

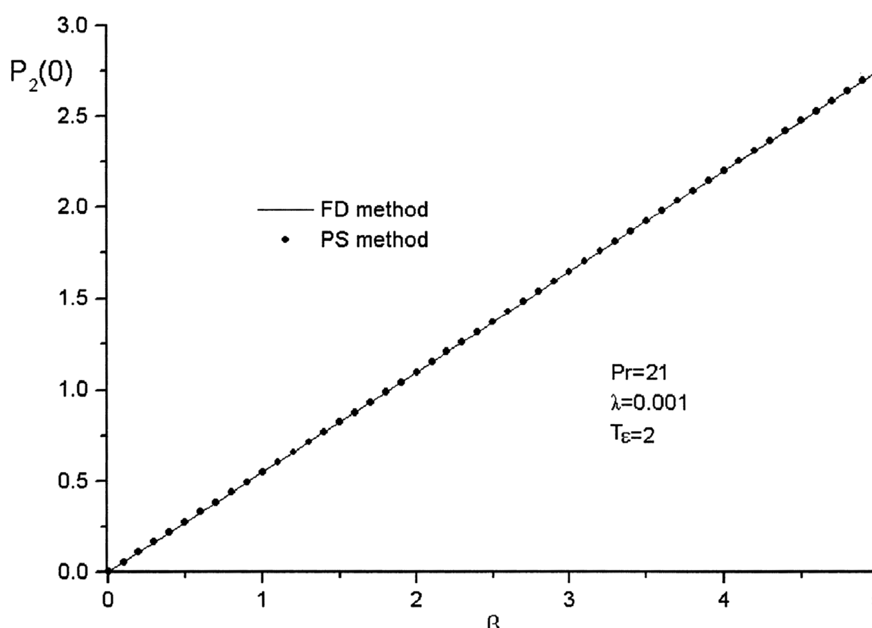


**Figure 4.**
Variation of the wall pressure parameter, $P_2(0)$

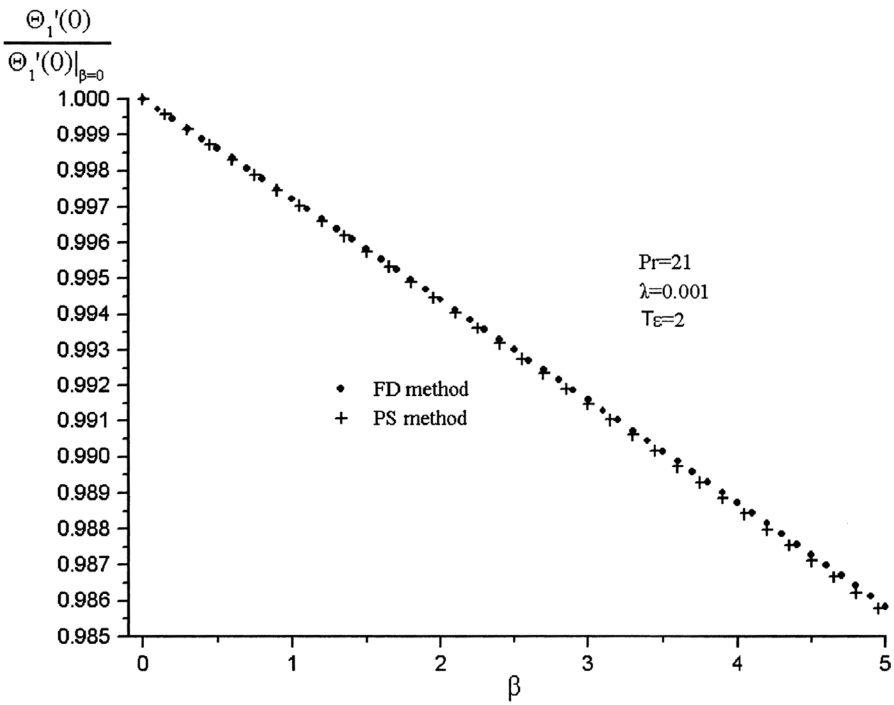


**Figure 5.**
Variation of the wall heat transfer parameter, $\Theta^*(0)$

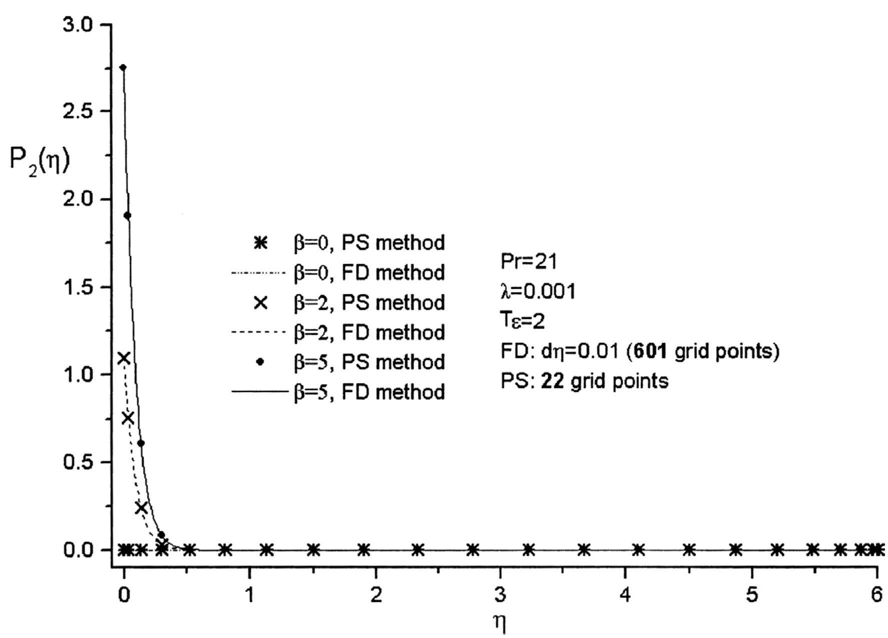


**Figure 6.**
Variation of the dimensionless pressure, $P_2(\eta)$

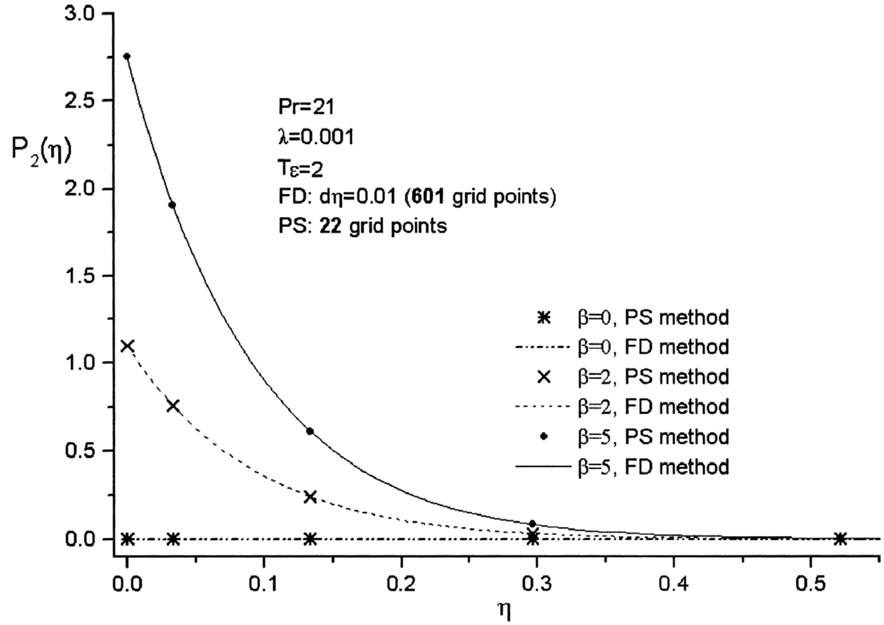


**Figure 7.**
Variation of the dimensionless pressure, $P_2(\eta)$, close to the boundary

Deeper analysis of the physical quantities of the problem under consideration, as well as the effect on the results of adopting a different function of magnetization are demonstrated in Tzirtzilakis and Kafoussias (2003). The main scope of this study is the investigation of the efficiency of the used PS method.

So, in order to compare the two methods, calculations were performed, with a PC Intel Pentium III at 750 MHz with 256 Mb RAM, for the wall shear parameter $-f''(0)$, the wall heat transfer parameter $\Theta'(0)$ and the heat transfer coefficient at the wall $\Theta^*(0)$. The value of tolerance, $\varepsilon$, between iterations was equal to $10^{-7}$. For the FD method, we used various step sizes $\Delta\eta$ from 0.2 to $5\times10^{-6}$, while for the CPSM, different number of grid points $N$ varying from 10 to 112. The values of the coefficients $-f''(0)$, $P_2(0)$ and $\Theta^*(0)$ are obtained for $\beta = 5$ and listed in Tables I and II for the FDs and PS method, respectively.

It is easy to observe from Tables I and II that accuracy of seven decimal places is achieved, in the values of the calculated coefficients mentioned earlier, after increasing or decreasing sufficiently $N$ or $\Delta\eta$ for the PS and FDs method, respectively. Thus, we can define as an error, the absolute difference of the value of coefficient achieved with seven decimal places from the value estimated for an $N$ or $\Delta\eta$. The values of the coefficients achieved with accuracy of seven decimal places that we consider are 1.4509820 for $-f''(0)$, 2.7538268 for $P_2(0)$ and 0.98581329 for $\Theta^*(0)$.

The variation of the error, as defined earlier, with the number of grid points used for the calculation of $-f^{//}(0)$ and $P_2(0)$, for both methods, is shown in Figure 8. It can be observed that there is a tremendous difference between the two methods, regarding the grid points they require in order to obtain the solution of a specific accuracy. It should be remarked here that in order to obtain the values of accuracy close to seven digits almost all the available RAM memory of the PC was used. Thus, for memory capacity reasons, we could not solve the system of equations under consideration obtaining much more accuracy than seven digits. Use of virtual memory in this case would be meaningless because of the great increment of computational time. Consequently, the PS method is memory minimizing and will be useful especially for multi-dimensional problems appearing in fluid mechanics. Figure 9 shows the reduction of error in the estimation of $-f^{//}(0)$, $P_2(0)$ and $\Theta^*(0)$ as the number of grid points $N$ increases in CPSM.

Finally, the variation of error in the calculation of $-f^{//}(0)$ and $P_2(0)$ with the CPU time, is shown schematically in Figure 10. It is apparent that the FD method requires less computational time for relatively great values of error. However, as error reduces, the FD requires more computational time and at error $\approx 0.001$ the two methods are equivalent. For smaller values of error the CPSM requires less computational time. For values of error less than $1 \times 10^{-4}$

| $-f^{//}(0)$ | $P_2(0)$ | $\Theta^*(0)$ | $\Delta\eta$ | CPU Time |
|---|---|---|---|---|
| 1.32225004 | 2.97639270 | 0.98781385 | 0.1 | 0.14 |
| 1.41180171 | 2.80990118 | 0.98639687 | 0.05 | 0.31 |
| 1.44919099 | 2.75607552 | 0.98584331 | 0.01 | 1.40 |
| 1.45052774 | 2.75438910 | 0.98582107 | 0.005 | 2.77 |
| 1.45096370 | 2.75384939 | 0.98581361 | 0.001 | 24.27 |
| 1.45098186 | 2.75382711 | 0.98581330 | 0.0001 | 356.75 |
| 1.45098200 | 2.75382694 | 0.98581329 | 0.00005 | 758.45 |
| 1.45098204 | 2.75382690 | 0.98581329 | 0.00002 | 1906.94 |
| 1.45098205 | 2.75382689 | 0.98581329 | 0.00001 | 3840.57 |

**Table I.** Variation of coefficients with the reduction of step size $\Delta\eta$ using FDs method

| $-f^{//}(0)$ | $P_2(0)$ | $\Theta^*(0)$ | $N$ | CPU Time |
|---|---|---|---|---|
| 1.32610607 | 2.38586163 | 0.98837892 | 11 | 0.91 |
| 1.45100053 | 2.75360337 | 0.98564635 | 22 | 2.67 |
| 1.45095901 | 2.75377334 | 0.98579618 | 25 | 3.36 |
| 1.45097791 | 2.75380780 | 0.98582949 | 28 | 5.43 |
| 1.45098232 | 2.75382658 | 0.98581268 | 32 | 7.10 |
| 1.45098206 | 2.75382684 | 0.98581330 | 42 | 11.87 |
| 1.45098207 | 2.75382682 | 0.98581329 | 46 | 14.30 |
| 1.45098200 | 2.75382688 | 0.98581329 | 112 | 126.51 |

**Table II.** Variation of coefficients with the increment of the number of grid points $N$ using PS method

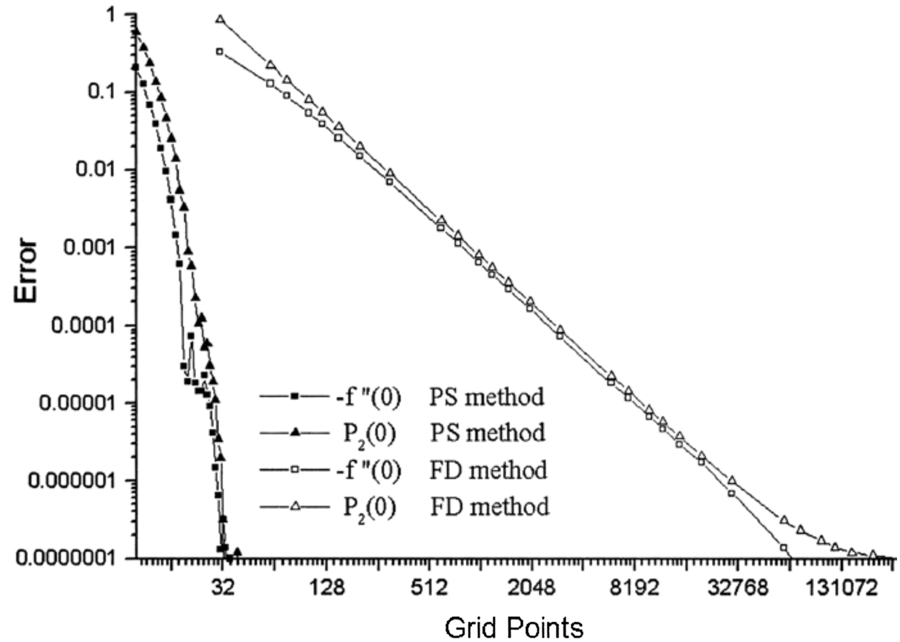


Figure 8.
Variation of the error with the grid points

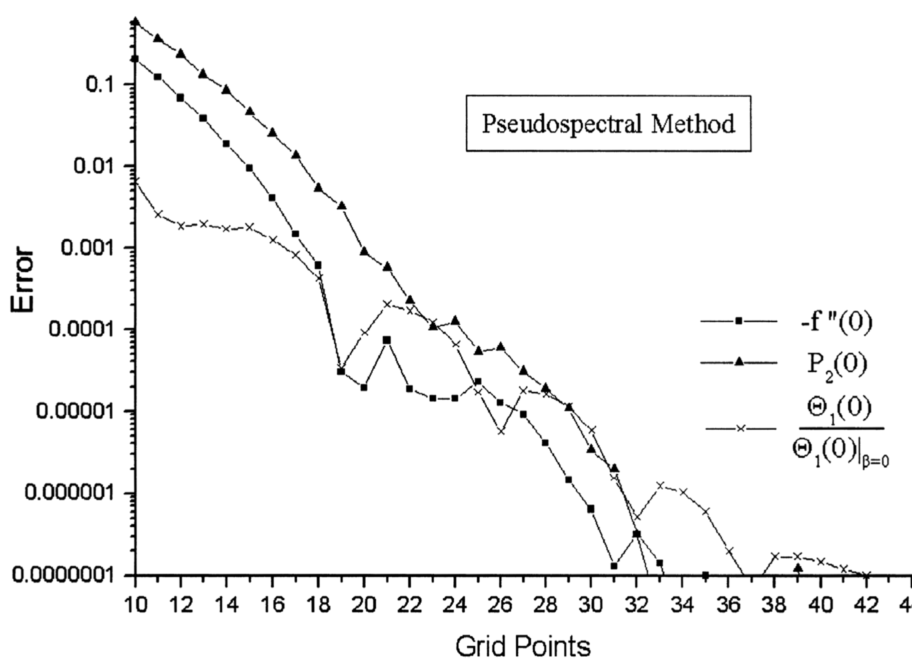


Figure 9.
Variation of the error with the grid points for the CPSM

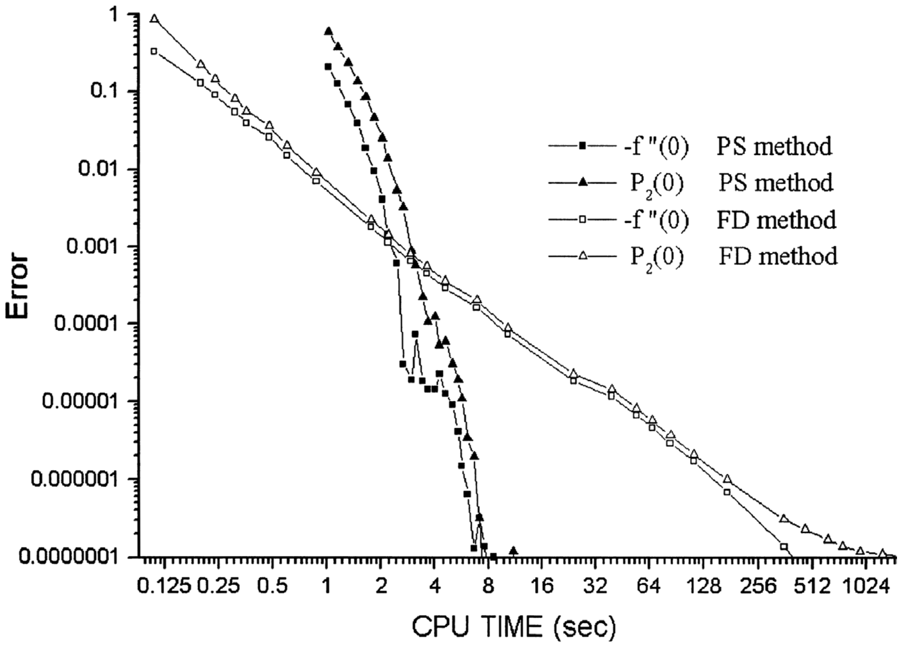


**Figure 10.**
Variation of the error with the CPU time

the CPU time required for FD is much more than that required for CPSM and finally, for values of order $1 \times 10^{-7}$, it is three orders of magnitude greater.

## Concluding remarks

In this study, we demonstrate and apply a CPSM in order to solve the coupled, non-linear system of ordinary differential equations, that arises from the mathematical formulation of a problem of fluid mechanics. We compare this method with a method based on the common FDs by calculating some characteristic coefficients of the physical problem. The results show that the CPSM is much faster and requires less memory than that of the FDs, if accuracy of more than three decimal places is desired. For example, for accuracy of six decimal places, FD method requires 30,000 grid points and 173.33 s CPU time while the CPSM requires only 32 grid points and 7.1 s CPU time. For lower values of accuracy, the FD method is slightly faster. The results denote that the CPSM could be very useful in obtaining the numerical solutions especially for multi-dimensional problems arising in fluid mechanics.

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