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Soliton-like solutions of higher order wave equations of the Korteweg–de Vries type

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In this work we study second and third order approximations of water wave equations of the Korteweg–de Vries (KdV) type. First we derive analytical expressions for solitary wave solutions for some special sets of parameters of the equations. Remarkably enough, in all these approximations, the form of the solitary wave and its amplitude-velocity dependence are identical to the sech^2 formula of the one-soliton solution of the KdV. Next we carry out a detailed numerical study of these solutions using a Fourier pseudospectral method combined with a finite-difference scheme, in parameter regions where soliton-like behavior is observed. In these regions, we find solitary waves which are stable and behave like solitons in the sense that they remain virtually unchanged under time evolution and mutual interaction. In general, these solutions sustain small oscillations in the form of radiation waves (trailing the solitary wave) and may still be regarded as stable, provided these radiation waves do not exceed a numerical stability threshold. Instability occurs at high enough wave speeds, when these oscillations exceed the stability threshold already at the outset, and manifests itself as a sudden increase of these oscillations followed by a blowup of the wave after relatively short time intervals.
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I. INTRODUCTION

As is well known, the Korteweg–de Vries (KdV) equation represents a first order approximation in the study of long wavelength, small amplitude waves of inviscid and incompressible fluids. Furthermore, if one allows the appearance of higher order terms, more complicated wave equations can be obtained. Such an equation, including second and third order corrections, was proposed in Ref. 1 and was examined, in its second order form, analytically and numerically in Refs. 2, 3, and 4. It was found that, although it is nonintegrable in general, it still possesses solitary wave solutions, which, for small values of parameters, behave like pure solitons.

One problem mentioned in Ref. 4 was that the solitary waves of this second order equation generally possess a nonzero background and thus might be unphysical.

In this work, we study in more detail this second order equation, as well as its third order counterpart proposed in Ref. 1, as approximations for water wave propagation. We first apply the Pickering algorithm[5,6] and introduce an additional arbitrary constant, which allows us to construct zero background solitary waves for both of these equations. Thus we demonstrate the remarkable fact that all these solutions have the same sech^2 form and the same amplitude dependence on the velocity as the one-soliton solution of the KdV.

We then proceed to conduct a numerical study and show that a range of parameters exists for which these solitary waves possess soliton-like behavior, in the sense that they interact nearly elastically with each other and are stable under small perturbations. We also demonstrate that all these results continue to hold in the case of the third order approximation of water wave propagation, for an even larger set of parameters.

Let us consider the famous KdV equation

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}+\alpha u\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}=0 \tag{1}$$

which constitutes a first approximation of unidirectional wave motion on the surface of a thin layer of an inviscid and incompressible fluid. The function $u(x,t)$ represents the amplitude of the fluid surface with respect to its level at rest, while α and β characterize, respectively, the long wavelength and short amplitude of the waves, compared with the depth of the layer.

In order to obtain a more physically realistic form of (1) one may include second order terms in α and β as suggested in Ref. 1

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}+\alpha u\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}+\alpha^2\rho_1 u^2\frac{\partial u}{\partial x}+\alpha\beta\left(\rho_2 u\frac{\partial^3 u}{\partial x^3}+\rho_3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\right)=0, \tag{2}$$

where ρ_1,ρ_2,ρ_3 are considered, for the time being, as free parameters. This equation holds for $\alpha,\beta\ll 1$, obeying $O(\beta)<O(\alpha)$, as, e.g., $\beta\approx\alpha^2$.[1] In Ref. 1 it was also observed that (2) can be transformed—up to terms of second order in α, β—to a completely integrable partial differential equation (pde), through a nonlinear local change of the dependent variable.

As mentioned in Refs. 2 and 3, Eq. (2) is, in general, nonintegrable in the sense that some of its ordinary differential equation (ode) reductions do not possess the Painlevé property and a Lax pair does not seem to exist. However, it was still found to possess the traveling wave solution:[3,4]

$$u(x,t)=K+\frac{3\beta k(A^2-2k)(2\rho_2+\rho_3)\mathrm{sech}^2[\sqrt{k}(x-Ct-x_0)/\sqrt{2}]}{\alpha\rho_1(A-\sqrt{2k}\tanh[\sqrt{k}(x-Ct-x_0)/\sqrt{2}])^2}, \tag{3}$$

where

$$K=\frac{2\rho_1-2\rho_2-\rho_3}{2\alpha\rho_1(\rho_2+\rho_3)}-\frac{\beta(2\rho_2+\rho_3)k}{\alpha\rho_1}, \tag{4}$$

$$C=\frac{4\rho_1-1}{4\rho_1}+\frac{(\rho_2-2\rho_1)^2}{4\rho_1(\rho_2+\rho_3)^2}+\frac{\beta^2\rho_3(2\rho_2+\rho_3)k^2}{\rho_1}, \tag{5}$$

and A,k,x_0 are arbitrary constants.

These waves were studied numerically in Ref. 4 and were found to possess, for small values of β and k, properties of true solitons: i.e., they are stable under small perturbations and interact elastically with each other. However, they also possess a generally nonzero ''background,'' given by (4), which means that they may be thought of as unphysical, since they have infinite energy (when integrated over the full real line).

As we show in this paper, however, this need not be true, since there are particular choices of the ρ_i parameters which make $K=0$ and thus restore to the solitary wave (3) its proper physical meaning. To establish this we use a method due to Pickering[5,6] and introduce an extra free parameter which helps us choose the ρ_i so that (3) finally becomes identical to the sech^2-profile of the KdV one-soliton solution.

Entirely analogous results are obtained if we allow third order terms in (2) and study solitary wave solutions of the pde

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}+\alpha u\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}+\alpha^2\rho_1 u^2\frac{\partial u}{\partial x}+\alpha\beta\left(\rho_2 u\frac{\partial^3 u}{\partial x^3}+\rho_3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\right)$$
$$+\alpha^3\rho_4 u^3\frac{\partial u}{\partial x}+\alpha^2\beta\left(\rho_5 u^2\frac{\partial^3 u}{\partial x^3}+\rho_6 u\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+\rho_7\left(\frac{\partial u}{\partial x}\right)^3\right)=0, \tag{6}$$

where $\rho_1,\ldots,\rho_7$ are again considered free parameters. This equation is also found to be valid for $O(\beta)<O(\alpha)$,[1] as e.g., $\beta\approx\alpha^2$, with $0<\alpha\leqslant 1$.

In fact, as we show in Sec. II, (6) possesses a solitary wave that has exactly the sech^2-form of the KdV soliton, for certain choices of the values of the parameters $\rho_1,\ldots,\rho_7$. These solitary wave solutions are obtained at ρ_i values which are different than the ones needed to derive (6) from the pair of pdes of bidirectional wave propagation given by first principles.[1] However, this does not mean that solitary waves cannot be found by other analytical or numerical methods and for other parameter values than those identified in this paper.

We then proceed, in Sec. III, to carry out a detailed numerical investigation of the stability of our solutions, using a combination of a Fourier pseudospectral method in space and a finite difference scheme in time with various step sizes Δt. Establishing first as maximum tolerance for numerical stability, $E=(\Delta t)^2$, we regard a solitary wave as stable if the small radiation waves, occurring due to numerical errors, do not exceed in amplitude this threshold.

Thus, we find regions of parameters for which such stable solutions exist, exhibiting small oscillations that remain bounded for all times. However, when the amplitude of these oscillations exceeds E they are seen to exhibit a sharp increase after relatively short times, leading eventually to blowup of the wave. In Sec. III we also study the interaction of three such stable solitary waves and show that they remain unchanged before and after collision, demonstrating thus their soliton-like character. Finally in Sec. IV we summarize our conclusions and list some open questions for future investigation.

II. ANALYTICAL EXPRESSIONS OF SOLITARY WAVE SOLUTIONS

In order to obtain explicit expressions for solitary wave solutions we shall employ Pickering's algorithm,[5,6] which was also used in Ref. 3 for the derivation of the solution (3)–(5). As can be seen in (4), however, for specific values of $\rho_i\alpha$ and β, K becomes zero only for one value of k. This means that such solutions would exist only for one particular velocity, which is inconsistent with what one finds for the soliton solutions of the KdV. It is possible, however, to obtain a zero background for a wider set of k values by introducing an additional arbitrary constant in (4), as follows:

If we consider a truncated expansion of the solution of (2) of the form

$$u(x,t)=\frac{u_0}{z^2}+\frac{u_1}{z}+u_2+u_3z+u_4z^2, \tag{7}$$

where the u_i's are constants and $z=z(x,t)$ satisfies the equations

$$\begin{aligned} z_x&=1-Az-Bz^2,\\ z_t&=-C+ACz+BCz^2 \end{aligned} \tag{8}$$

with A, B, and C also free constants, we can allow one of the u_i to be arbitrary. This happens, for example, if

$$\rho_1=0 \quad \text{and} \quad \rho_3=-2\rho_2, \tag{9}$$

in which case u_0 is arbitrary and

$$u_1=-Au_0, \quad u_2=\frac{1}{12}(A^2-8B)u_0-\frac{au_0+12b}{12ab\rho_2}, \quad u_3=u_4=0.$$

Substituting relation (7) in (2) and using (8) we finally obtain

$$u(x,t)=K-\frac{B_1(A^2-4B_1)u_0\,\mathrm{sech}^2[\sqrt{B_1}(x-Ct-x_0)]}{(A-2\sqrt{B_1}\tanh[\sqrt{B_1}(x-Ct-x_0)])^2}$$

(where we have set $B_1=B+\frac{1}{4}A^2$) with

$$K=-\frac{1}{\alpha\rho_2}+\frac{1}{12}\left(4B_1-\frac{1}{\beta\rho_2}\right)u_0,$$

$$C=\frac{\rho_2-1}{\rho_2}-\frac{\alpha u_0}{12\beta\rho_2}+\frac{4}{3}\alpha\beta B_1^2\rho_2 u_0,$$

x_0 being the arbitrary location of the "center" of the wave. We can now force the background to be zero ($K=0$) by choosing

$$u_0=\frac{12\beta}{\alpha(4\beta\rho_2B_1-1)} \tag{10}$$

(hence u_0 is no longer arbitrary) and conclude with the solution

$$R3U_0:\ u(x,t)=-\frac{B_1(A^2-4B_1)u_0\,\mathrm{sech}^2[\sqrt{B_1}(x-Ct-x_0)]}{(A-2\sqrt{B_1}\tanh[\sqrt{B_1}(x-Ct-x_0)])^2}, \tag{11}$$

where

$$C=1+4\beta B_1$$

is the velocity of the traveling wave.

Observe that (11) can in fact be written in the form of the well-known sech^2-soliton solution of the KdV (1) by a simple transformation: Writing

$$\cosh\theta=\frac{A}{\sqrt{A^2-4B_1}}\quad\text{and}\quad\sinh\theta=\frac{2\sqrt{B_1}}{\sqrt{A^2-4B_1}}\quad\text{for } A^2>4B_1$$

and shifting x_0 appropriately, (11) is easily seen to take the form

$$u(x,t)=-\frac{3(C-1)}{\alpha(-1+(C-1)\rho_2)}\,\mathrm{sech}^2\left[\frac{1}{2}\sqrt{\frac{C-1}{\beta}}(x-Ct-x_0)\right]$$

which is exactly the one-soliton solution of the KdV if $\rho_2=0$ in which case (2) reduces exactly to (1).

In the expansion (7) of the Pickering algorithm we may alternatively consider u_2 as an arbitrary constant, by setting

$$\rho_2=2\rho_1\quad\text{and}\quad\rho_3=-2\rho_1 \tag{12}$$

and thus obtain explicit solutions of (3) even in the case $\rho_1\neq 0$. Conditions (12) then lead to the solution

$$u(x,t)=K+\frac{12\beta B_1(A^2-4B_1)\mathrm{sech}^2[\sqrt{B_1}(x-Ct-x_0)]}{\alpha(A-2\sqrt{B_1}\tanh[\sqrt{B_1}(x-Ct-x_0)])^2},$$

where

$$K=\frac{3\beta(A^2-4B_1)}{\alpha}+u_2,$$

$$C=1+3A^2\beta-8\beta B_1+\alpha u_2+\rho_1(3A^2\beta-12\beta B_1+\alpha u_2)(3A^2\beta-4\beta B_1+\alpha u_2),$$

and A,B_1,x_0 are again arbitrary constants. Zero background is obtained by setting

$$u_2=-\frac{3\beta(A^2-4B_1)}{\alpha}, \tag{13}$$

whence we arrive at the expression

$$R3U2:\ \ u(x,t)=\frac{12\beta B_1(A^2-4B_1)\mathrm{sech}^2[\sqrt{B_1}(x-Ct-x_0)]}{\alpha(A-2\sqrt{B_1}\,\tanh[\sqrt{B_1}(x-Ct-x_0)])^2}, \tag{14}$$

where the velocity of the wave is again $C=1+4\beta B_1$.

It is worth remarking here that if, instead of applying Pickering's approach, we were to consider the traveling wave reductions of (2), $u(x,t)=f(x-Ct)$, for the choice of parameters (12), we can integrate the resulting ode and discover, by a simple phase plane analysis, that it possesses a separatrix along which the solution is

$$u(x,t)=\frac{3(C-1)}{\alpha}\,\mathrm{sech}^2\left[\frac{1}{2}\sqrt{\frac{C-1}{\beta}}(x-Ct-x_0)\right]. \tag{15}$$

This is exactly the same as the sech^2-soliton solution of KdV (1) for all ρ_1 and also coincides with (14), for $C=1+4\beta B_1$ if we shift x_0 appropriately, as explained below (11).

Finally, let us turn to the third order equation (6). Here it is important to point out that a traveling wave reduction and a derivation of the solitary wave form as done above appears to be quite difficult, as the associated odes are too cumbersome to integrate exactly. Thus we need to turn to the application of Pickering's algorithm and show, as before, that u_0, A, and B remain arbitrary iff ρ_i satisfy the following relations:

$$\rho_3=2(\rho_1-\rho_2),\quad \rho_4=0,\quad \rho_5=2\rho_1(\rho_2-2\rho_1),$$
$$\rho_6=6\rho_1(2\rho_1-\rho_2),\quad \rho_7=3\rho_1(\rho_2-2\rho_1), \tag{16}$$

whence the corresponding solution takes the form

$$u(x,t)=K-\frac{B_1(A^2-4B_1)u_0\,\mathrm{sech}^2[\sqrt{B_1}(x-Ct-x_0)]}{(A-2\sqrt{B_1}\,\tanh[\sqrt{B_1}(x-Ct-x_0)])^2},$$

where K and C depend on the parameters of the equation and the arbitrary constants u_0, A, and B_1. The zero background solution $(K=0)$ arises if we set

$$u_0=\frac{12\beta}{\alpha(4\beta B_1\rho_2-8\beta B_1\rho_1-1)}, \tag{17}$$

whence we finally obtain

$$R7U0:\ \ u(x,t)=-\frac{B_1(A^2-4B_1)u_0\,\mathrm{sech}^2[\sqrt{B_1}(x-Ct-x_0)]}{(A-2\sqrt{B_1}\,\tanh[\sqrt{B_1}(x-Ct-x_0)])^2} \tag{18}$$

with $C=1+4\beta B_1$ again the velocity of the wave. Note that, with $\rho_1=0$, solution (18) with (17) coincides exactly with (11) and (10).

In the same way, as with the second order equations, we may also consider u_2 arbitrary and derive the following ρ_i relations:

$$\begin{gathered}\rho_2=2\rho_1, \quad \rho_3=-2\rho_1, \quad \rho_5=3\rho_4,\\ \rho_6=-6\rho_4, \quad \rho_7=3\rho_4,\end{gathered} \tag{19}$$

whence the corresponding solution is

$$u(x,t)=K+\frac{12\beta B_1(A^2-4B_1)\operatorname{sech}^2[\sqrt{B_1}(x-Ct-x_0)]}{\alpha(A-2\sqrt{B_1}\tanh[\sqrt{B_1}(x-Ct-x_0)])^2},$$

where K and C depend again on the parameters of the equation and the arbitrary constants u_2, A and B_1. The zero background solution $(K=0)$ now requires

$$u_2=-\frac{3\beta(A^2-4B_1)}{\alpha} \tag{20}$$

and we finally obtain

$$R7U2\colon\ u(x,t)=\frac{12\beta B_1(A^2-4B_1)\operatorname{sech}^2[\sqrt{B_1}(x-Ct-x_0)]}{\alpha(A-2\sqrt{B_1}\tanh[\sqrt{B_1}(x-Ct-x_0)])^2}, \tag{21}$$

where the velocity is $C=1+4\beta B_1$, as in all the cases above. Again here (21) also becomes identical to the KdV soliton (15), with the appropriate shift of the constant x_0, even though it is the solution of a much more complicated pde.

It is important, however, to remark that the ρ_i parameter values, (16) or (19), determined by our approach, are quite different from the ones found in Ref. 1, by the reduction to unidirectional flow from a pair of pdes describing bidirectional wave propagation. The reasons for this difference remains an open question, which clearly requires further investigation.

III. NUMERICAL STABILITY ANALYSIS

The numerical scheme used in the current study is the same as the one employed in Ref. 4 and is based on a combination of finite differences and a Fourier pseudospectral method. In order to demonstrate the application of our algorithm we first describe it on the KdV equation

$$u_t+u_x+\alpha uu_x+\beta u_{xxx}=0 \tag{22}$$

with the initial condition $u(x,0)=f(x)$. The time derivative in (22) is discretized using a finite difference approximation, in terms of central differences

$$u^{n+1}=u^{n-1}-2\Delta t(u_x^n+\alpha u^n u_x^n+\beta u_{xxx}^n)=0. \tag{23}$$

According to the pseudospectral method, we introduce the approximate solution

$$u(x,t)=\sum_{k=0}^{N}\alpha_k(t)\Phi_k(x), \tag{24}$$

where $\Phi_k(x)=e^{ikx}$ are the Fourier exponentials, and $\alpha_k(t)$ are coefficients to be determined, for $k=0,1,...,N$.

The steps used to advance the solution from time step n to $n+1$ are[7]

(i) Given $u_j^n=u(x_j,t_n)$ evaluate $\alpha_k^n=\alpha_k(t_n)$ from (24).
(ii) Given α_k^n evaluate the derivatives, e.g., $[\partial^2u/\partial x^2]_j^n$ from (24).

(iii) Evaluate the nonlinear terms, e.g., $u_j^n[\partial u/\partial x]_j^n$.
(iv) Evaluate u_j^{n+1} from (23), at $x=x_j$, $t=t_{n+1}$.

Step (i) is the transformation from physical space to spectral space. This transformation is achieved by the use of a fast Fourier transform (FFT) described in Refs. 7 and 8 with a number of operations $(5/2)N\log_2 N$ (N being the number of polynomials), in contrast to the $2N^2$ operations required for a matrix-vector multiplication.[9] Step (ii) occurs in spectral space and the evaluation of the nonlinear term in step (iii) is in physical space, thus avoiding the expensive multiplication of all coefficients in the expansions of the form (24). Step (iv) occurs again in physical space.

The accuracy of our numerical scheme for the time variable t is $O((\Delta t)^2)$, due to central differences and for the space variable x, where we use the pseudospectral method, $O(e^{-qN})$, where q is a constant.[8] Numerical calculations were carried out for various numbers of polynomials $N=128$, 256, 512, and 1024 and time steps $\Delta t=0.0001$ to 0.002, while the spatial step was chosen to be $\Delta x=1$.

We should mention here that, for the time propagation of such types of problems, where the spatial discretization is extremely accurate, the most commonly used method is the fourth order Runke–Kutta integration scheme. Even though this method provides satisfactory results, it may fail because of sensitivity to the initial conditions and inherent instabilities. Thus, since the stability of the waves propagating in time is of more interest than the accuracy, a more stable, central differencing is used for the discretization in time.

In Ref. 4 we carried out several calculations to verify the efficiency of our numerical code. For the KdV equation (1) at $t=0$ with $\alpha=1$, $\beta=0.1$, $x_0=20$ and $c=1.1$, we took as initial condition the well-known exact solitary wave solution

$$u(x,t)=\frac{3(c-1)}{\alpha}\,\mathrm{sech}^2\left[\frac{1}{2}\sqrt{\frac{c-1}{\beta}}\,(x-ct-x_0)\right], \tag{25}$$

where c is the propagation speed and x_0 is an arbitrary constant.[10,11] We observed that our wave moves along the spatial direction retaining its initial profile for a very long time period of at least $t=2.5\times10^6$ time units with time step $\Delta t=0.01$. A three-soliton interaction was also studied and the results were as expected from the soliton solutions of the KdV, i.e., the waves interact elastically and remain unchanged before and after their interaction. These results were obtained for various time steps and numbers of polynomials N mentioned above, which demonstrates that our code reproduces accurately the fundamental properties of the KdV.

The plethora of free parameters entering into Eqs. (2) and (6) makes the study of the wave solutions, obtained in Sec. II, not a very easy task. However, if we impose the zero background for our solutions, much of the redundancy is removed and our ρ_i's begin to have a more specific meaning. Thus, we investigate the wave solution (11) for Eq. (2), with u_0 given by (10) and for ρ_i, $i=1,2,3$ satisfying (9). This solution is referred to as $R3U0$. For the same equation we also study the solution (14) for u_2 given by (13) and ρ_i, $i=1,2,3$ satisfying (12), which is referred to as $R3U2$. Similarly we name $R7U0$ the solution (18) of Eq. (6) with u_0 given by (17) and for ρ_i, $i=1,...,7$ satisfying (16) and $R7U2$ the solution (21) of the same equation with u_2 given by (20) and for ρ_i, $i=1,...,7$ satisfying (19).

The free parameters now present in the solutions $R3U0$ and $R3U2$ are only α, β, B_1, and ρ_2 for $R3U0$ or ρ_1 for $R3U2$. Therefore, we will first study how they affect the stability of the above-mentioned wave solutions, and then proceed to study the $R7U0$ and $R7U2$ waves, using similar α and β, plus ρ_1 for $R7U0$ and ρ_4 for $R7U2$.

A. A stability criterion

Our ultimate goal, of course, is to examine the values of the parameters in our higher order KdV equations (2) and (6), for which the solitary wave solutions mentioned above preserve their shape and are stable under evolution. By the term "stable" we mean that a wave solution, when substituted in an equation, retains its initial profile for long times, albeit with some smaller oscillations present as radiation waves, due to unavoidable numerical errors produced under time evolution.

Thus, in order to check stability, one way is to track the residual of the solution in time. For the case of KdV, for example, if $\overline{u}$ is an exact solution of (1) it will satisfy

$$\overline{u}_t+\overline{u}_x+\alpha\overline{u}\overline{u}_x+\beta\overline{u}_{xxx}=0. \tag{26}$$

If the approximate solution (24), computed numerically, is substituted into (26) it will not, of course, give zero. Thus we write for it

$$u_t+u_x+\alpha uu_x+\beta u_{xxx}=R,$$

where R is called the residual of the equation. It is expected that R is a continuous function of x and t and if N is sufficiently large then, in principle, the coefficients $\alpha_k(t)$ can be chosen so that R is as small as we wish over the computational domain. In our case we evaluate $R=R_i$ at each x_i, $i=1,...,N$ grid point at specific time moments t_n.

Due to the fact that the wave solutions are computed for sufficiently large values of N (128 to 1024), the spatial error of the pseudospectral method is in agreement with the $O(e^{-qN})$ estimate mentioned above, and is practically zero. The maximum absolute residual, which we refer to as the error, $E=\max_i|R_i|$, will increase due to the central differencing in time, but cannot be greater than $O((\Delta t)^2)$. Several tests have been made for the wave solution (25) of the KdV verifying that for various values of N (128 to 1024) and time step $\Delta t=0.0001$ to 0.02, $E<(\Delta t)^2$ at least for a time period of 10^6 time units.

Therefore, a practical way to verify that a wave solution is stable is to check if the error remains, for long times, less than $O((\Delta t)^2)$. If E increases above this value already from the outset, oscillations will soon grow and become unbounded after relatively short times, not only because of the numerical scheme, but also due to the nonlinear nature of the equations, suggesting that the initial wave solution has become unstable. *This is also supported by the fact that blowup occurs nearly at the same times, irrespective of the values of the Δx and Δt step sizes used in the numerical scheme.*

B. Solutions $R3U0$ and $R3U2$

Let us now proceed to the study of $R3$ solutions. In all that follows B_1 satisfies the relations $B_1>0$ and $A>\sqrt{2B_1}$ which are vital in order to have a bounded wave solution. The parameter A does not affect the stability of the wave and in all cases is taken $A=3$.

One way to examine the stability of solutions under investigation is to set α, β fixed and start to increase B_1, the velocity of the wave, by a quantity ΔB_1. Each time we increase B_1, we track the error E for a period of time: If it remains below $(\Delta t)^2$, we consider that the wave is stable for this period, and proceed to increase B_1 by another ΔB_1 until E becomes greater than $(\Delta t)^2$. Once this happens, we set $\Delta B_1=\Delta B_1/2$, decrease B_1 by ΔB_1 and track the error again. In that way, we determine, up to an accuracy ϵ ($\Delta B_1<\varepsilon$), the maximum value of B_1 (speed), $B_1^{\max}$, for which the solitary wave is stable.

The period of time used in the current study is 500 time units, with time step $\Delta t=0.001$ and accuracy $\varepsilon=0.001$. Several tests have been made, e.g., with the KdV, using cases where the exact solution is known to be stable, and as expected, the value of $B_1^{\max}$, estimated in the above way, depends neither on the time step Δt, nor on the number of points N we use.

Figure 1 shows the variation of $B_1^{\max}$ with α for various values of β. The parameter ρ_2 is fixed equal to 1. It is observed that as α increases $B_1^{\max}$ is increasing for small values of β. The variation of $B_1^{\max}$ with α is smaller as β increases and finally no significant changes are observed when $\beta>0.2$. Thus, it can be concluded that α has a stabilizing effect on the solutions, especially for low

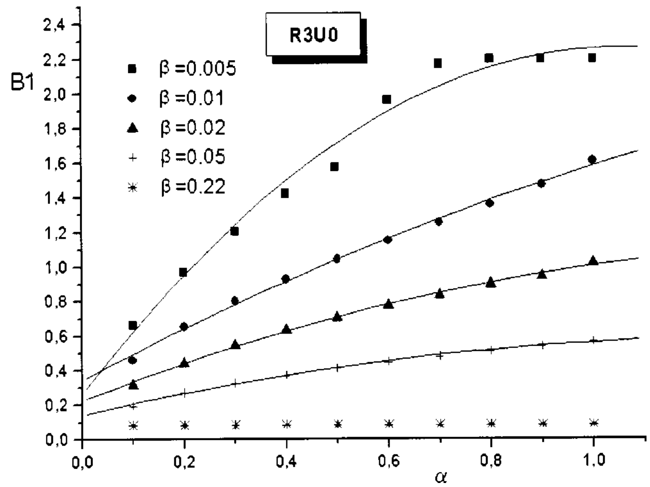

FIG. 1. Variation of $B_1^{\max}$ of solution $R3U0$ with α.

values of β, since its increase makes the range of velocities larger for stable soliton-like waves (in the range of $0<\beta\leqslant 0.1$). The region below each curve (plotted by interpolation) is the region of stability of the wave solution for the corresponding values of parameters.

The variation of the $B_1^{\max}$ with β for various values of α is shown in Fig. 2. Unlike what was observed for α in Fig. 1, we find that $B_1^{\max}$ decays exponentially with the increase of β, and the region where the wave solution is stable is larger for greater values of α. Thus we conclude that increasing β has a destabilizing effect on the solitary waves. The parameter ρ_2 is again kept equal to 1. The variation of $B_1^{\max}$ for the solution $R3U2$ with α or β is qualitatively the same as that for $R3U0$, and any quantitative differences with Figs. 1 and 2 are insignificant.

Similar results were also observed for the $R7U0$ and $R7U2$ solutions. However, because of the additional ρ_i parameters present in these cases, a direct comparison with the $R3$ solutions is not easy to demonstrate pictorially.

It is important to note that these findings are in good agreement with the conditions of the validity of (2) and (6), i.e., that $O(\beta)<O(\alpha)$.[1] In fact, using $\beta\approx\alpha^2$ is seen to yield optimal results in terms of the size of stability regions of our solitary wave solutions.

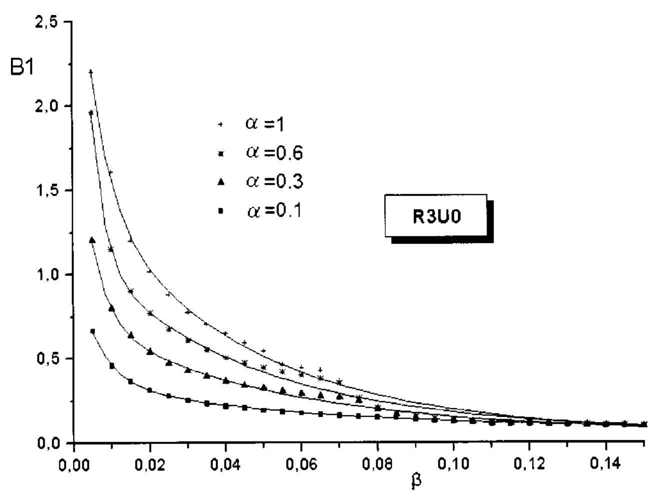

FIG. 2. Variation of $B_1^{\max}$ of solution $R3U0$ with β.

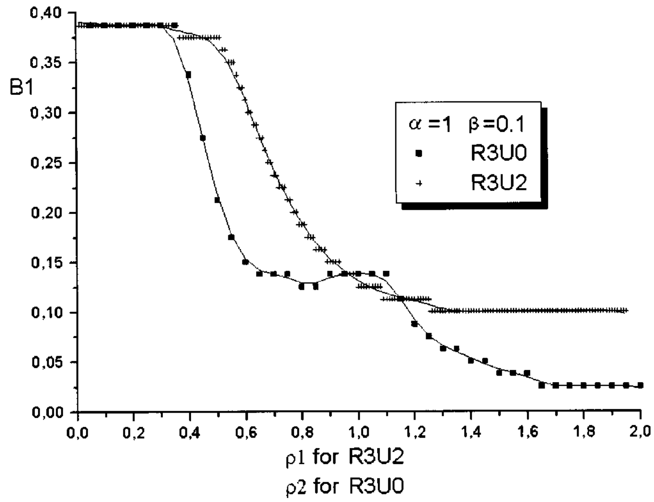

FIG. 3. Variation of $B_1^{\max}$ with ρ_2 for $R3U0$ and ρ_1 for $R3U2$.

In order to examine the effect of ρ_2 on $R3U0$ or ρ_1 on $R3U2$ we can set $\alpha=1$, $\beta=0.1$ and find $B_1^{\max}$ while varying the corresponding ρ_i. Figure 3 shows this variation of $B_1^{\max}$ with ρ_2 for $R3U0$ and ρ_1 for $R3U2$ solitary wave solutions. It is found that for an increase of the corresponding ρ_i up to 0.4 there are no significant changes. When the ρ_i increase beyond 0.4 a rapid decrease of $B_1^{\max}$ takes place. This decrease stops at $\rho_1 \approx 1.2$ for the $R3U2$ solution, while, in the case of $R3U0$ it continues until ρ_2 reaches the value of approximately 1.7.

It is worth mentioning that the results plotted in Figs. 1–3 using $\Delta t=0.001$ and $N=128$, are also obtained for time steps 0.0001, 0.002, and number of points 256 and 512.

C. Error behavior during wave propagation

Before proceeding to the $R7$ solutions, let us discuss some results concerning the error criterion of Sec. III A, in order to understand how wave solutions propagate in time and which of them are considered as stable. The results described below were obtained for the $R7U0$ solution, but are similar to what is observed for $R7U2$.

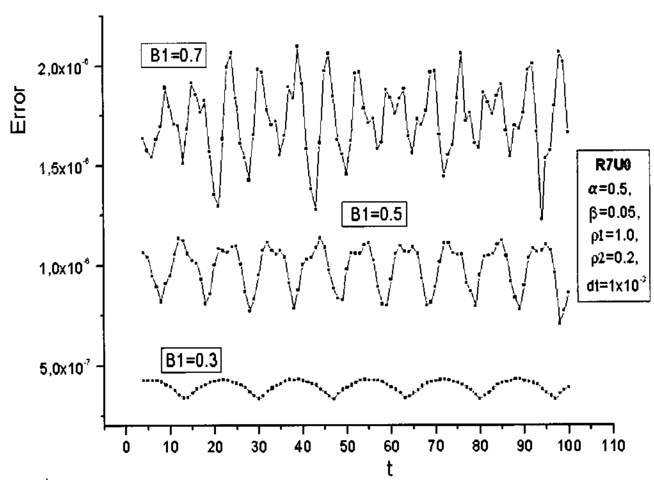

FIG. 4. Error variation with time for three different values of B_1. The case $B_1=0.5$ lies just above the boundary of the stability region $E \leqslant (\Delta t)^2$.

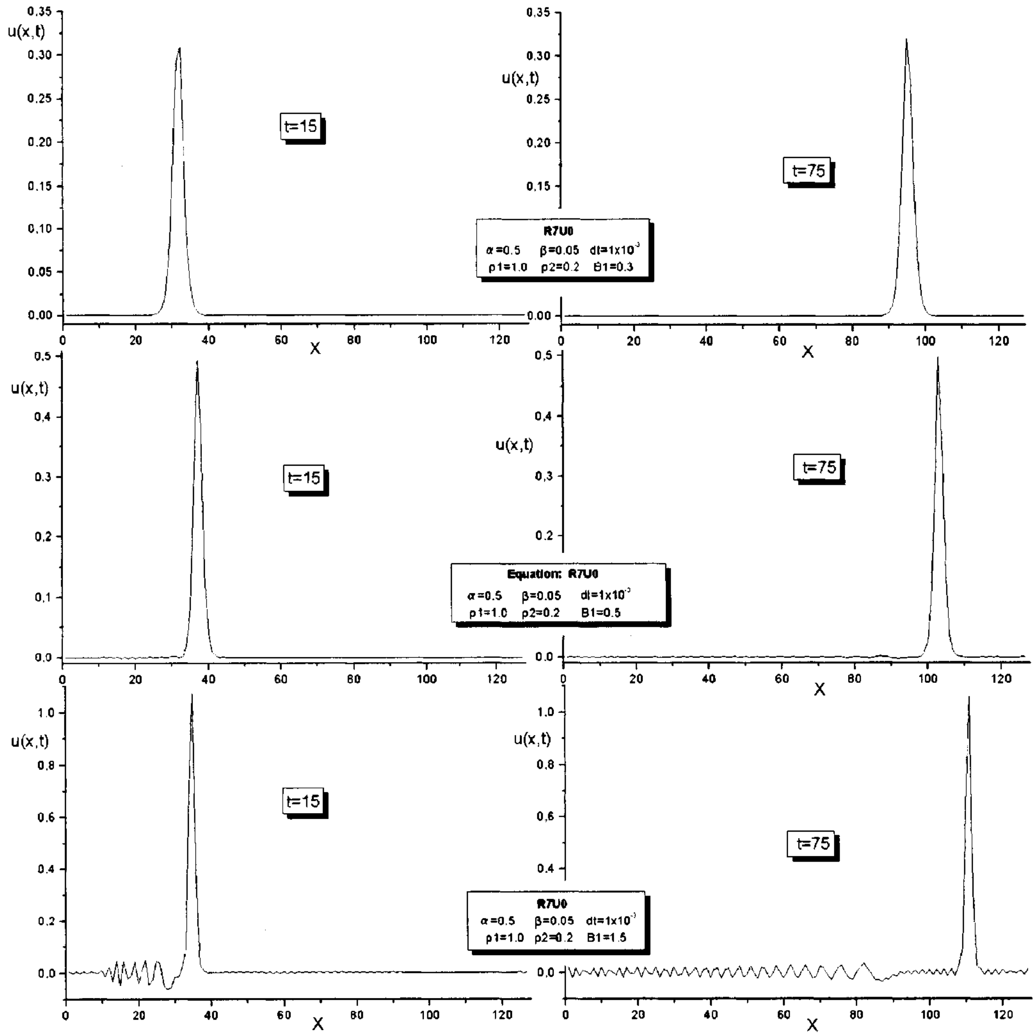

FIG. 5. Evolution of $R7U0$ in time for $B_1 = 0.3$, 0.5, and 1.5, respectively.

We set $\alpha = 0.5$, $\beta = 0.05$, $\rho_1 = 1$, $\rho_2 = 0.2$ and track the error of $R7U0$ for a period of time of 100 time units, with time step $\Delta t = 0.001$, and three different values of B_1, namely 0.3, 0.5, and 0.7. The variation of the errors for these cases are shown in Fig. 4. Note that the error for $B_1 = 0.3$ is less than $(\Delta t)^2 = 1 \times 10^{-6}$.

Figure 5 shows the $R7U0$ solution propagating in time for B_1 equal to 0.3, 0.5 and 1.5, respectively, for the same values of the other parameters as above. It is observed that for $B_1 = 0.3$ the wave remains virtually unchanged in time and is therefore considered stable, with its radiation waves remaining smaller than $(\Delta t)^2$ for very long times. On the other hand, in the case $B_1 = 0.5$, bounded oscillations appear where the error slightly exceeds $(\Delta t)^2$. We call this solution unstable, because its radiation waves grow as time increases and become unbounded after a relatively short time. Similarly, for $B_1 = 1.5$, where these oscillations are even larger at the beginning, blowup occurs after a much shorter time interval. We remark once more that analogous results are obtained for different time steps, and also for the same time step and greater values of N.

It is important to mention that for small $\beta(\beta < 0.1)$ and large $\alpha(\alpha \approx 1)$ oscillations appear, even when the error E does not exceed $(\Delta t)^2$. These oscillations remain almost the same for a time period comparable to the one used to test the stability of the solutions of KdV. Moreover,

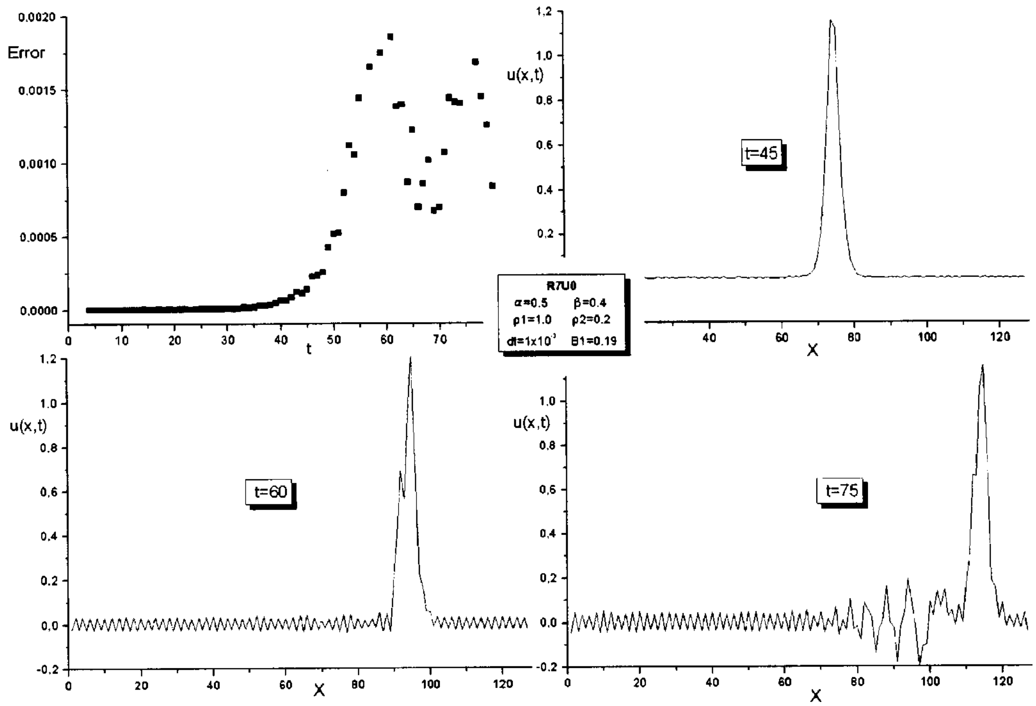

FIG. 6. Evolution of $R7U0$ in time for $B_1=0.19$.

even when $E\approx(\Delta t)^2$ (and β small enough) oscillations can persist over long time intervals as seen in Fig. 5, with $\beta=0.05$, where the oscillations occurring for $B_1=1.5$ remain unchanged and bounded well beyond value of $t=75$ shown in the figure.

However, if the error grows sharply at some point in time, this implies that the oscillation will become unbounded soon thereafter leading to a blowup of the solution. Figure 6 shows the propagation of $R7U0$ for $\alpha=0.5$, $\beta=0.4$, $\rho_1=1$ and $\rho_2=0.2$. In this case, $B_1^{\max}\approx 0.156$, as estimated by the method described above. Consequently, if we set $B_1=0.19$ error oscillations suddenly explode at $t\approx 50$ causing the wave amplitude to increase while at $t\approx 80$ the solution blows up. The same behavior is observed for Δt varying from 0.0001 to 0.002 and N from 128 to 512 indicating that this is not a numerical blowup.

As in the case of $R7U0$ shown in Fig. 6, we have also observed from numerous tests, that if the wave is to blow up, the error will suddenly increase by 2–3 orders of magnitude within the first 200 time units. In some examples, blowup occurs after thousands of time units, but is predicted by our error analysis already within the first 200 time units. Thus, for the calculation of stability results, we adopt the time period of 500 time units of numerical integration.

In order to investigate the stability of the $R7$ solutions we keep the values of α and β fixed at $\alpha=1$, $\beta=0.1$. Moreover, we consider $\rho_1=0.2$ and $\rho_2=0.2$ for $R7U2$ and $R7U0$, respectively. Consequently, the independent parameters are ρ_1 for $R7U0$ and ρ_4 for $R7U2$. The variation of $B_1^{\max}$ with ρ_1 for $R7U0$ and ρ_4 with $R7U2$ is shown in Fig. 7. It is found that the growth of ρ_1 in $R7U0$ results in an increase of $B_1^{\max}$, whereas increasing ρ_4 in $R7U2$ solution results in a decrease of $B_1^{\max}$. For relatively large values of the corresponding ρ_i the stability regions differ considerably and for values greater than unity the corresponding $B_1^{\max}$ can be 4 times greater for $R7U0$ than that of $R7U2$.

D. Elastic three wave interactions

In Ref. 4 a three-wave interaction was performed with Eq. (2), using its solution (3) as initial condition for the three solitary waves. It was reported that due to the different backgrounds of the

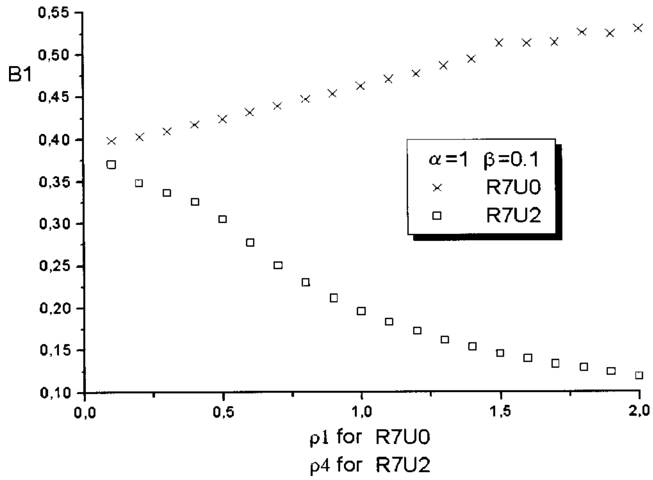

FIG. 7. Variation of the $B_1^{\max}$ with ρ_1, for $R7U0$ and ρ_4 for $R7U2$.

waves a slight displacement of each solution by a constant had to be applied. This can now be avoided using the zero background wave solutions we have obtained here. Thus, we study here the interactions of solitary waves of the form (15), using the third order KdV equation (6).

A preliminary investigation of the $R7$ solutions suggests using the $R7U0$ solitary wave, since its stability region increases with increasing ρ_1, provided we keep the difference between ρ_1 and ρ_2 small. The reason is that the additional nonlinear terms in $R7U0$ are multiplied by $\rho_4,\ldots,\rho_7$ values which are smaller than those of $R7U2$, and thus lead to a larger area of stability.

The next step is to specify values for the α and β parameters. As was shown in Sec. II, the wave speed of propagation is $C=1+4\beta B_1$. Thus, the relative speed between two waves is $4\beta(B_1-B_1')$, where $C'=1+4\beta B_1'$ is the velocity of the second wave. Consequently, we have to use relatively large values of β to see an interaction within a reasonable time. Furthermore, we have to avoid using large α so as to reduce the bounded oscillations described in the preceding section. Therefore, taking into account all these considerations we choose $\alpha=0.4$, $\beta=0.1$, $\rho_1=0.25$, and $\rho_2=0.1$.

For the values of the remaining parameters we set $x_1=10$, $x_2=28$, $x_3=55$ and $B_{11}=0.30$, $B_{12}=0.20$, $B_{13}=0.06$, where x_i and B_{1i} correspond, for $i=1,2,3$, to the first (fastest), second (middle), and third (slowest) wave. The interaction of these solitary waves, as shown in Fig. 8, is seen to occur in exactly the same way as for the KdV equation. No radiation is observed and no differences are found in the shape of the solitary waves (before and after collision) as far as we could determine numerically (with $\Delta t=0.01$ and $N=1024$). These results strongly indicate the existence of wave solutions which "behave" as true solitons in water wave equations which represent higher order approximations to the KdV equation.

IV. CONCLUDING REMARKS

In this paper, we have studied the existence and stability of solitary wave solutions of pdes representing second and third order approximations of unidirectional water wave propagation, in the short amplitude, $0<\alpha=a/h\ll 1$, and long wavelength limit, $0<\beta=(h/l)^2\ll 1$ (h is the depth of the fluid layer). To first order in α and β, these pdes reduce to the famous KdV equation, which is completely integrable and possesses solitary waves that interact perfectly elastically with each other and are called solitons.

Our original motivation was the fact that these higher order KdV equations have been shown to be equivalent to completely integrable pdes, by a local nonlinear transformation, at the same order of approximation in α and β.[1] The question therefore naturally arises if these higher order KdV approximations also possess solitary waves exhibiting soliton-like dynamics.

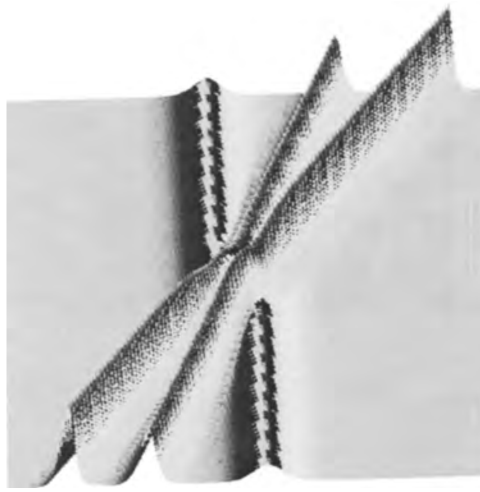

FIG. 8. Elastic interaction of three $R7U0$ solitary waves of Eq. (6).

Adopting the assumption $O(\alpha) < O(\beta)$ (e.g., $\beta \approx \alpha^2$), which eliminates some dispersive terms with higher order derivatives,[1,12] we derive exact, closed form expressions for the solitary waves valid to second and third order in α and β using Pickering's algorithm.[5,6] Choosing then specific values for the free parameters, we force these solutions to have zero background and demonstrate the remarkable fact that they all have the same sech^2 form and velocity dependence as the simple, one-soliton solution of the KdV.

Proceeding to a numerical study of their stability, we use a Fourier pseudospectral method combined with finite difference in time (with step size Δt) and establish a threshold of numerical error tolerance $E = (\Delta t)^2$. Thus we call a solitary wave stable if the small oscillations trailing the wave have amplitude smaller than E and remain bounded for very long times ($\leqslant 500$ units).

However, as the speed of the wave increases, these ''radiation'' oscillations also increase and when their amplitude exceeds E, already at the beginning of their evolution, turn out to exhibit a dramatic growth over relatively short times ($\leqslant 200$ units) leading eventually to blowup of the wave and characterizing the solution as unstable. Our results are entirely consistent with what is observed for the KdV equation and also agree with the assumption that $O(\beta) < O(\alpha)$ in these pdes.

A number of open questions remain for future investigation: What is the ''physical'' meaning of the values of the free parameters, which we have chosen so that the solitary waves of these pdes have zero background? Why are these values different from those obtained by the reduction of the original water wave equations to unidirectional motion?[1] In fact, we have recently observed that the ode that gives traveling wave solutions for the second order approximation (2) can be easily integrated once and then numerically studied by phase plane analysis to give homoclinic orbits, which correspond to solitary waves for many other ρ_i parameters than the ones identified here.

Can such integrations be carried out also for the third order equation (6) to yield solitary waves, exhibiting similar behavior and reducing to the KdV soliton, as α and β go to zero? Finally, among all these choices, which ρ_i fit best the physical realization of a solitary water wave?

Another question concerns the mathematical form of the solitary waves we have obtained in this paper: Could their simple sech^2 expressions (identical for the second and third order case) imply that they also hold in higher order approximations? Finally, is their presence related to the fact that the corresponding approximation can be transformed to a pde which is completely integrable to the same order in α, β?

In conclusion, we believe that water wave motion still remains an open and fascinating topic of great mathematical and physical interest and hope to be able to answer some of the above questions in a future publication.

ACKNOWLEDGMENTS

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