Z. angew. Math. Phys. 54 (2003) 551–565
0044-2275/03/040551-15
DOI 10.1007/s00033-003-1100-5

Zeitschrift für angewandte Mathematik und Physik ZAMP

Biomagnetic fluid flow over a stretching sheet with non linear temperature dependent magnetization

E. E. Tzirtzilakis and N. G. Kafoussias

Abstract. The flow of a heated ferrofluid over a linearly stretching sheet is studied in the presence of an applied magnetic field due to a magnetic dipole. It is assumed that the applied magnetic field is sufficiently strong to saturate the ferrofluid and the variation of magnetization with temperature can be approximated by a non linear function of temperature difference. By introducing appropriate non dimensional variables the problem is described by a coupled and non linear system of ordinary differential equations with its boundary conditions which is solved numerically by applying an efficient numerical technique based on the common finite difference method. The obtained results are presented graphically for different values of the parameters entering into the problem under consideration and the dependence of the flow field from these parameters is discussed. A comparative study, with a similar problem which has already been solved and documented in literature, is also made wherever necessary, emphasizing the importance of the non-linear variation of magnetization with temperature. Emphasis is also given in the obtained results for Prandtl number equal to 21 and critical exponent $\delta = 0.368$ which are important and interesting in Biomagnetic Fluid Dynamics.

Mathematics Subject Classification (2000). $76D05$, $34A34$, $76M20$, $76W05$, $76Z99$

Keywords. Biomagnetic fluid flow, magnetic dipole, magnetization, numerical solution.

1. Introduction

A ferromagnetic fluid consists of a stable colloidal dispersion of subdomain magnetic particles in a liquid carrier. The properties of the ferrofluid are profoundly affected by the thermal Brownian motion of the suspended particles and the fact that each subdomain particle is permanently magnetized [1]. A particularly attractive feature of the ferrofluids is the dependence of the magnetization upon the temperature and this thermomagnetic coupling makes ferrofluids useful in various practical applications. So, in recent years an extensive work has been done on the ferrofluid dynamics in the presence of magnetic field [2]-[6].

The behavior of a ferrofluid, under the action of an applied magnetic field, is also of fundamental importance in the development of Biomagnetic Fluid Dynamics (BFD) in which the blood is investigated as a magnetic fluid. It was found that blood can be considered as a magnetic fluid because the red blood cells contain the hemoglobin molecule, which is a form of iron oxides present at a uniquely high concentration in the mature red blood cells. So, blood possesses the property of a magnetic material, and under some circumstances, can be considered as diamagnetic, paramagnetic or ferromagnetic fluid [7]. In order to examine the flow of a biomagnetic fluid under the action of an applied magnetic field, Haik et al. [7] developed a mathematical model for the Biomagnetic Fluid Dynamics (BFD). BFD differs from MagnetoHydroDynamics (MHD) in that it deals with no electric current and the flow is affected by the magnetization of the fluid in the magnetic field. In MHD, which deals with conducting fluids, the mathematical model ignores the effect of polarization and magnetization. The behavior of a biomagnetic fluid, when it is exposed to magnetic field (magnetized), is described by the magnetization property M. Magnetization is the measure of how much the magnetic field is affecting the magnetic fluid and, in general, is a function of the magnetic field intensity H and the temperature T.

The two classical problems in fluid mechanics, namely the Blasius boundary layer flow along a flat plate and the stagnation point flow, were extended for a saturated ferrofluid under the combined influence of thermal and magnetic field gradients by Neuringer [8]. The flow of a viscous fluid past a linearly stretching surface in otherwise quiescent surroundings was first considered by Crane [9] for a Newtonian fluid and subsequently extended to fluids obeying non-Newtonian constitutive equations like viscoelastic [10], micropolar [11] and inelastic power-law fluids [12]. Some of these cases were later extended to include the effect of a uniform transverse magnetic field on the motion of an electrically conducting fluid driven by the stretching sheet [13]-[15].

Andersson and Valnes [2] extended Crane's problem by studying the influence of the magnetic field, due to a magnetic dipole, on a shear-driven motion (flow over a stretching sheet) of a viscous non-conducting ferrofluid. The fluid flow was formulated as a five-parameter problem, and the influence of the magneto-thermo-mechanical coupling explored numerically. It was concluded that the primary effect of the magnetic field was to decelerate the fluid motion as compared to the hydrodynamic case. In their study, they also considered that the magneto-thermo-mechanical coupling is completely described by assuming that the applied magnetic field H is sufficiently strong to saturate the ferrofluid and the variation of magnetization M with temperature T can be approximated by the linear equation of state $M = K\,(T_c - T)$.

However, Arrot et al. [16] suggested that below Curie temperature T_c, the variation of spontaneous magnetization M, with temperature T, of the magnetic particles in a liquid carrier, is given by the expression

$$M = M_1 \left(\frac{T_c - T}{T_1} \right)^{\delta},$$

where δ is the critical exponent for the spontaneous magnetization and M_1, T_1 are constants dependent on the material of the magnetic particles.

So, in the present work the flow of a heated ferrofluid over a stretching sheet, in the presence of an applied magnetic field due to a magnetic dipole, is studied as in [2]. It is assumed that the magneto-thermo-mechanical coupling is not described by a linear function of temperature difference, as in [2], but by the above mentioned expression suggested by Arrot et al. [16]. The formulation of the problem is obtained by an analogous manner presented in [2] and the numerical solution is obtained by applying an efficient numerical technique based on the common finite difference method [17]. The obtained results are presented graphically, for different values of the parameters entering into the problem under consideration, and the dependence of the flow field on these parameters is also discussed. A comparative study with the Andersson and Valnes' problem [2] is also made wherever necessary, emphasizing the importance of the non-linear variation of magnetization with temperature.

2. Mathematical formulation

The steady, two–dimensional, incompressible, laminar biomagnetic fluid flow, past a flat and impermeable elastic sheet, which is stretched with a velocity proportional to the distance x $(u{=}cx)$, is shown schematically in Fig. 1. The biomagnetic fluid far away from the sheet is at rest and at temperature T_c, while the stretched sheet is kept at fixed temperature T_w, less than T_c. The viscous and electrically non-conducting fluid, is confined to the half space $(y > 0)$ above the sheet, whereas a magnetic dipole is located below it, at distance d.

The magnetic field of the dipole gives rise to a magnetic field of sufficient strength to saturate the biomagnetic fluid. The equations and the boundary conditions governing this flow model are the mass conservation, fluid momentum at x and y direction and energy equation and can be written as :

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu_0 M\frac{\partial H}{\partial x} + \mu\nabla^2 u, \tag{2}$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu_0 M\frac{\partial H}{\partial y} + \mu\nabla^2 v, \tag{3}$$

$$\rho c_p\, \vec{q}\cdot\left(\vec{\nabla}T\right) + \mu_0 T\frac{\partial M}{\partial T}\vec{q}\cdot\left(\vec{\nabla}H\right) = k\nabla^2 T + \mu\Phi, \tag{4}$$

with boundary conditions :

$$y = 0 \quad : \qquad u = cx, \quad v = 0, \qquad T = T_w \tag{5}$$

$$y \to \infty \quad : \qquad u = 0, \qquad T = T_c, \quad p + 1/2q^2 = const. \tag{6}$$

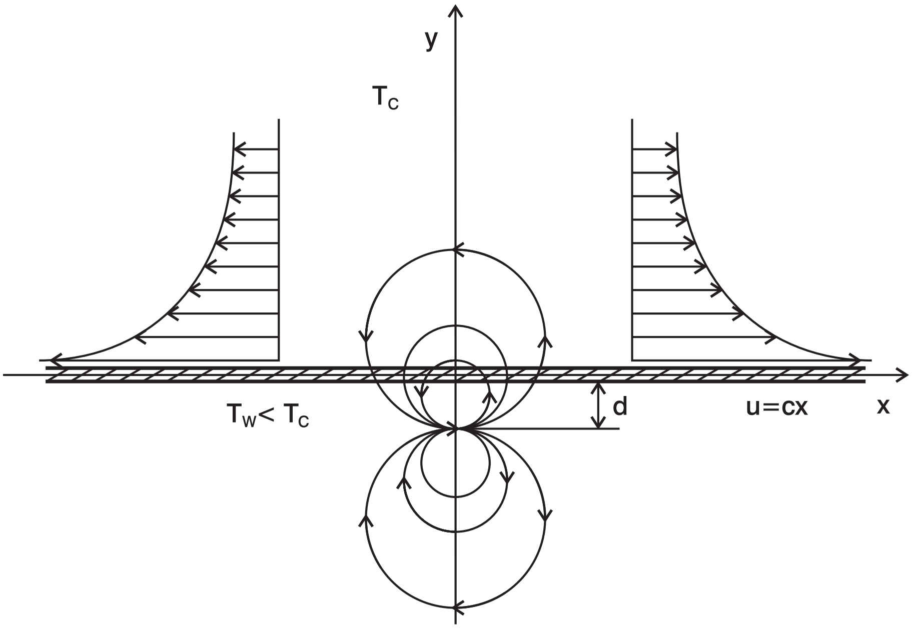

Figure 1. Schematic representation of flow configuration

In the above equations u and v are the velocity components of the fluid in x and y direction, respectively ($\vec{q} = (u, v)$), p the pressure, ρ the biomagnetic fluid density, μ the dynamic viscosity, μ_o the magnetic permeability, c_p the specific heat at constant pressure and k the thermal conductivity. Also ∇^2 is the two dimensional Laplacian operator ($\nabla^2 = \vec{\nabla} \cdot \vec{\nabla} = \partial^2 / \partial x^2 + \partial^2 / \partial y^2$) and Φ is the dissipation function which, in our case, is given by

$$\Phi = 2 \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right] + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 . \tag{7}$$

The terms $\mu_o M (\partial H / \partial x)$ and $\mu_o M (\partial H / \partial y)$ in (2) and (3), respectively, represent the components of the magnetic force per unit volume and depend on the existence of the magnetic gradient. When the magnetic gradient is absent these forces vanish. The second term, on the left-hand side of the thermal energy equation (4), accounts for heating due to adiabatic magnetization.

Following [2], we consider that the components H_x , H_y of the magnetic field $\vec{H} = (H_x, H_y)$, due to a magnetic dipole, are given by

$$H_x\left(x, y\right) = -\frac{\partial V}{\partial x} = \frac{\gamma}{2\pi} \, \frac{x^2 - (y+d)^2}{\left[x^2 + (y+d)^2 \right]^2} \, , \tag{8}$$

$$H_y\left(x,y\right)=-\frac{\partial V}{\partial y}=\frac{\gamma}{2\pi}\,\frac{2x\left(y+d\right)}{\left[x^2+\left(y+d\right)^2\right]^2}\ , \tag{9}$$

where

$$V\left(x,y\right)=\frac{\alpha}{2\pi}\,\frac{x}{x^2+\left(y+d\right)^2} \tag{10}$$

is the scalar potential of the magnetic dipole, $\gamma=\alpha$ and α is a dimensionless distance defined as $\alpha=(c\rho/\mu)^{1/2}\,d$. Thus the magnitude $\left\|\vec{H}\right\|=H,$ of the magnetic field intensity, is given by

$$H\left(x,y\right)=\left[H_x^2+H_y^2\right]^{1/2}=\frac{\gamma}{2\pi}\,\frac{1}{x^2+\left(y+d\right)^2} \tag{11}$$

and the gradients are given by

$$\frac{\partial H}{\partial x}=-\frac{\gamma}{2\pi}\,\frac{2x}{\left(y+d\right)^4}\ , \qquad \frac{\partial H}{\partial y}=\frac{\gamma}{2\pi}\left[-\frac{2}{\left(y+d\right)^3}+\frac{4x^2}{\left(y+d\right)^5}\right]. \tag{12}$$

When the applied magnetic field $\vec{H}$ is sufficiently strong to saturate the biomagnetic fluid, the magnetization M is, generally, determined by the fluid temperature. Andersson and Valnes [2] considered that the variation of magnetization M with temperature T can be approximated by the linear equation of state $M=K\left(T_c-T\right)$, where K is a constant called pyromagnetic coefficient and T_c is the Curie temperature.

However, Arrot et al. [16] suggested that below Curie temperature T_c , the variation of spontaneous magnetization M, with temperature T, of the magnetic particles in a liquid carrier, is given by the expression

$$M=M_1\left(\frac{T_c-T}{T_1}\right)^{\delta}, \tag{13}$$

where δ is the critical exponent for the spontaneous magnetization and M_1 , T_1 are constants dependent on the material of the magnetic particles. For iron $\delta=0.368$, $M_1=54\ Oe$, $T_1=1.45\ {}^oK$ and $T_c=770\ {}^oK$.
It is worth mentioning here that equation (13), used in this work, is more general than that used in [2] and becomes identical to it when $\delta=1$ and $M_1/T_1=K$.

3. Transformation of equations

We introduce now the nondimensional coordinates [2]

$$\xi(x)=\left(c\frac{\rho}{\mu}\right)^{1/2}x\ , \qquad \eta(y)=\left(c\frac{\rho}{\mu}\right)^{1/2}y \tag{14}$$

and the dimensionless variables

$$\Psi(\xi,\eta) = \left(\frac{\mu}{\rho}\right)\xi f(\eta)\ , \tag{15}$$

$$P\left(\xi,\eta\right) = \frac{p}{c\mu} = -P_1\left(\eta\right) - \xi^2 P_2\left(\eta\right)\ , \tag{16}$$

$$\Theta\left(\xi,\eta\right) = \frac{T_c - T}{T_c - T_w} = \Theta_1\left(\eta\right) + \xi^2\Theta_2\left(\eta\right). \tag{17}$$

In the above expressions $\Psi(\xi,\eta)$, $\Theta(\xi,\eta)$ and $P(\xi,\eta)$ are the dimensionless stream function, temperature and pressure, respectively. The velocity components can be calculated now as

$$u = \frac{\partial\Psi}{\partial y} = cxf'\left(\eta\right) \qquad\qquad v = -\frac{\partial\Psi}{\partial x} = -\left(\frac{c\mu}{\rho}\right)^{1/2} f\left(\eta\right) \tag{18}$$

where $(\ )' = {}^{\partial(\ )}/_{\partial\eta}$.

Substituting equations (12) $\sim$ (18) into the momentum equations (2) and (3) and the energy equation (4) and equating coefficients of equal powers of ξ , up to ξ^2 , we get the following system of ordinary differential equations

$$f''' + ff'' - \left(f'\right)^2 + 2P_2 - \frac{2B\Theta_1^\delta}{\left(\eta+\alpha\right)^4} = 0\ , \tag{19}$$

$$P_1' - f'' - ff' - \frac{2B\Theta_1^\delta}{\left(\eta+\alpha\right)^3} = 0\ , \tag{20}$$

$$P_2' - \frac{2B\Theta_2}{\left(\eta+\alpha\right)^3}\delta\Theta_1^{\delta-1} + \frac{4B\Theta_1^\delta}{\left(\eta+\alpha\right)^5} = 0\ , \tag{21}$$

$$\Theta_1'' + \Pr f\ \Theta_1' + \frac{2\lambda B\left(\Theta_1-\varepsilon\right)\ f}{\left(\eta+\alpha\right)^3}\delta\ \Theta_1^{\delta-1} + 2\Theta_2 - 4\lambda\left(f'\right)^2 = 0\ , \tag{22}$$

$$\begin{aligned}&\Theta_2'' - \Pr\left(2f'\ \Theta_2 - f\ \Theta_2'\right) + \frac{2\lambda B\ f\ \Theta_2}{\left(\eta+\alpha\right)^3}\delta\ \Theta_1^{\delta-2}\ \left[\left(1-\delta\right)\varepsilon + \delta\Theta_1\right]\\ &\quad -\lambda B\left(\Theta_1-\varepsilon\right)\left[\frac{2f'}{\left(\eta+\alpha\right)^4} + \frac{4f}{\left(\eta+\alpha\right)^5}\right]\delta\Theta_1^{\delta-1} - \lambda\left(f''\right)^2 = 0.\end{aligned} \tag{23}$$

Also, the boundary conditions (5) and (6) become now

$$\eta = 0 \quad : \qquad f = 0, \quad f' = 1, \quad \Theta_1 = 1, \quad \Theta_2 = 0, \tag{24}$$

$$\eta \to \infty \quad : \qquad f' \to 0, \quad \Theta_1 \to 0, \quad \Theta_2 \to 0, \quad P_1 \to -P_\infty, \quad P_2 \to 0. \tag{25}$$

The five dimensionless parameters appearing in the transformed equations are

$$\left.\begin{array}{ll}
\Pr = \mu c_p/k & (\textit{Prandtl number}), \\
\lambda = \dfrac{c\mu^2}{\rho k\,(T_c - T_w)} & (\textit{viscous dissipation parameter}), \\
\varepsilon = T_c/\,(T_c - T_w) & (\textit{dimensionless Curie temperature}), \\
B = \dfrac{\gamma}{2\pi}\,\dfrac{\mu_o M_1\,(T_c - T_w)^{\delta}\,\rho}{\mu^2 T_1^{\delta}} & (\textit{ferromagnetic interaction , parameter}), \\
\alpha = (c\rho/\mu)^{1/2}\,d & (\textit{dimensionless distance } \alpha)
\end{array}\right\} \quad (26)$$

It is worth reminding, once more, that for $\delta = 1$ and $M_1/T_1 = K$ (pyromagnetic parameter) the system of equations (19)–(26) becomes identical to that presented in [2].

The system of equations (19)–(23), subjected to the boundary conditions (24)-(25), is a five–parameter coupled and non–linear system, describing the biomagnetic fluid flow over the stretching sheet when the magnetization of the fluid is given by a non–linear temperature dependent relation.

4. Numerical method and results

One of the most commonly used algorithms for the solution of such two–point boundary value problems is the Runge–Kutta integration scheme along with the Newton–Raphson shooting method and such a technique was used by Andersson and Valnes in their study [2]. Even though this method provides satisfactory results for such types of problems, it may fail when applied to problems in which the differential equations are very sensitive to the choice of the missing initial conditions. Moreover, another serious difficulty which may be encountered in boundary–value problems is the inherent instability. Difficulty also arises in the case in which one end of the range of integration is at infinity. The end-point of integration is usually approximated by assigning a finite value to this point; it is obtained by estimating a value at which the solution will reach its asymptotic state. The computing time for integrating the differential equations can sometimes depend critically on quality of the initial guesses of the unknown boundary conditions and the infinite end-point.

On the contrary to the above–mentioned numerical method, the numerical technique we used in the present work has better stability characteristics, is simple, accurate and efficient. The essential features of this technique are the following: (i) It is based on the common finite difference method with central differencing, (ii) on a tridiagonal matrix manipulation, and (iii) on an iterative procedure. This technique is described in detail in [17].

So, numerical calculations were carried out for different values of the dimensionless parameters (26), entering into the problem under consideration and especially

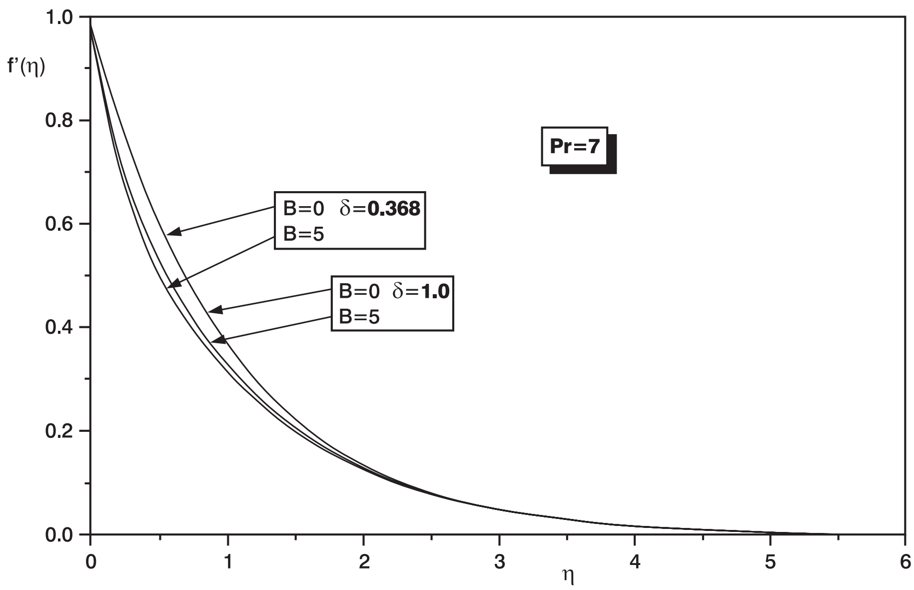

Figure 2. Variation of the dimensionless velocity component $f'(\eta)$

for different values of the Prandtl number Pr and critical exponent δ. It was stated in the introduction that blood, under some circumstances, can be considered as ferromagnetic fluid. Although the viscosity μ, the specific heat under constant pressure c_p and the thermal conductivity k of any fluid, and hence of the blood, are temperature dependent, Prandtl number $Pr = \mu c_p/k$ can be considered constant. Thus, for human body temperature $T = 310\ {}^oK$, the value of μ, c_p and k is equal to $3.2\times10^3 kgr/m{\cdot}sec$, $14.65\ Joule/kgr{\cdot}{}^oK$ and $2.2\times10^{-3} Joule/m{\cdot}sec{}^oK$, respectively, [18], [19] and hence $Pr = 21$.

In order to compare our results with those obtained in [2], numerical calculations were also carried out for Prandtl number $Pr = 1, 7,$ and for critical exponent $\delta = 1$. For this value of δ, our ferromagnetic interaction parameter B becomes identical to ferrohydrodynamic interaction parameter β in [2]. Hence, the values of B were taken from 0 to 5 as in [2]. The value $B = 0$ corresponds to hydrodynamic flow and in such a case the flow field is also independent on the critical exponent δ. Finally, the values of the viscous dissipation parameter λ, dimensionless Curie temperature ε and dimensionless distance α were taken equal to 0.01, 2 and 1, respectively.

The variations of the dimensionless velocity component $f'(\eta)$ $(= u/cx)$ are presented in Fig. 2 for $Pr = 7$, for two different values of the critical exponent $\delta = 1.0$, 0.368 and for two different values of the ferromagnetic interaction parameter B. It is observed that as the ferromagnetic interaction parameter B is increased the fluid velocity is decreased and this reduction is greater for smaller values of

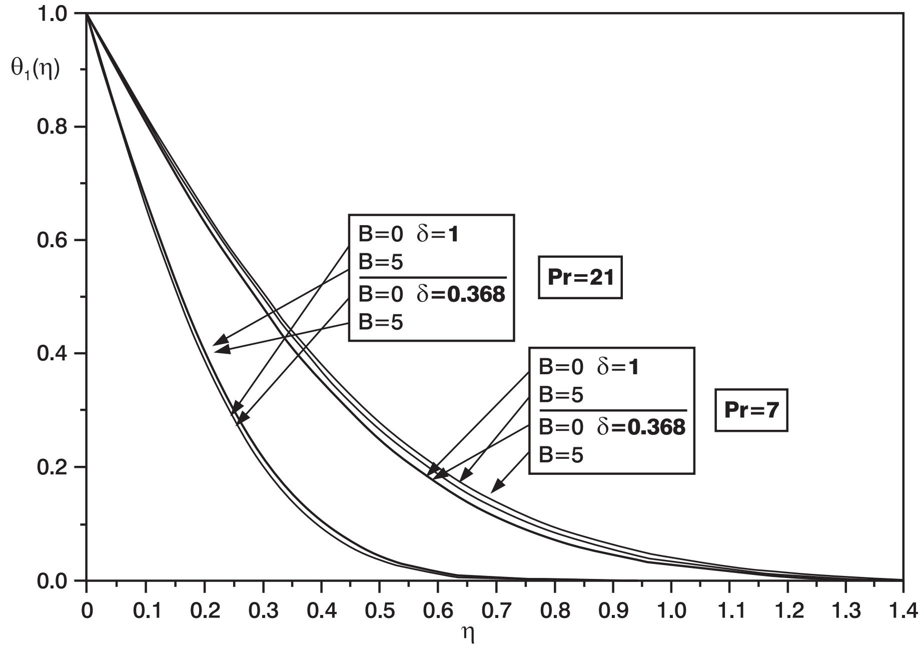

Figure 3. Variation of the dimensionless temperature $\Theta_1(\eta)$.

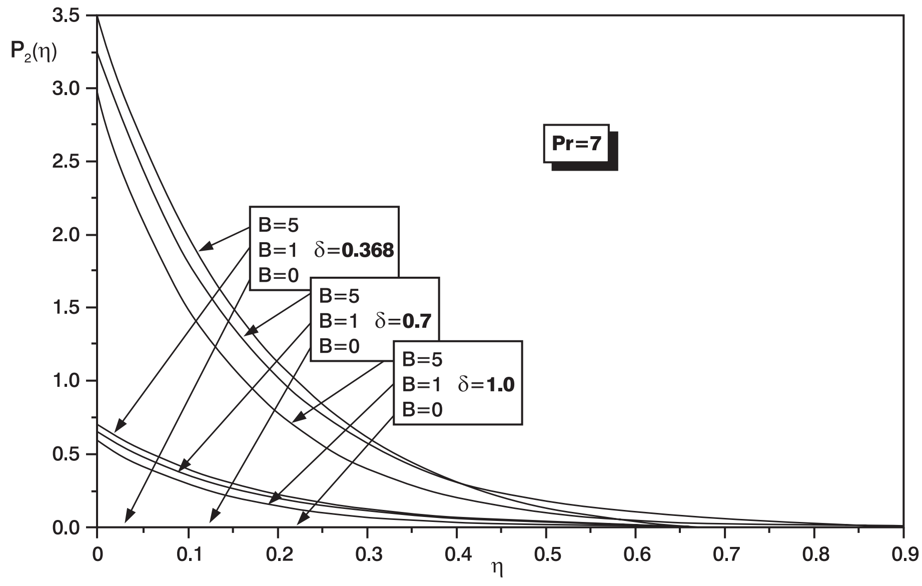

Figure 4. Variation of the dimensionless pressure $P_2(\eta)$.

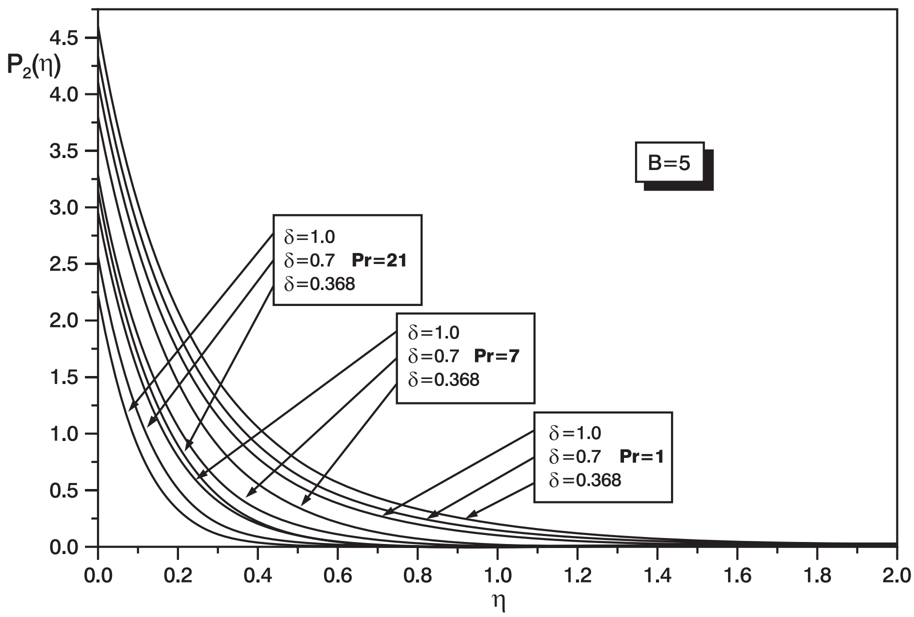

Figure 5. Variation of the dimensionless pressure $P_2(\eta)$.

the critical exponent δ . This influence of the above mentioned parameters on the velocity field, is limited for small values of the dimensionless distance η from the stretched sheet. The results for $\delta = 1$ and B =0 or 5 are identical to those obtained in [2].

Fig. 3 shows some characteristic profiles for the dimensionless temperature $\Theta_1(\eta)$ for Pr =7 and Pr =21 and for the same values of the parameters B and δ as in Fig. 2. It is observed from this figure that as Pr increases $\Theta_1(\eta)$ decreases. When $Pr = 7$ and $B = 5$ the temperature Θ_1 increases as the critical exponent δ decreases. However, this increment of $\Theta_1(\eta)$ is negligible for higher values of Pr . Finally, for every value of Pr and δ the fluid temperature increases as the ferromagnetic interaction parameter B increases. This increment is greater for small values of Pr and/or δ . The temperature profiles $\Theta_1(\eta)$ for $\delta = 1$, $Pr = 1$, and $B = 0$ or 5, correspond to those obtained in [2].

Figs. 4 and 5 show the variations of the dimensionless pressure $P_2(\eta)$ for different values of Pr , δ , and B . For $Pr = 7$, $\delta = 1.0$ and $B = 0$, 1 and 5 (Fig. 4) the obtained results are identical to those obtained in [2]. It is observed from Fig. 4 that for every value of the ferromagnetic interaction parameter $B \neq 0$ the dimensionless pressure P_2 increases as the critical exponent δ decreases and this increment is greater for higher values of B.

It was reported that a ferromagnetic fluid consists of a stable colloidal dispersion of subdomain magnetic particles in a liquid carrier. Hence, every value of the

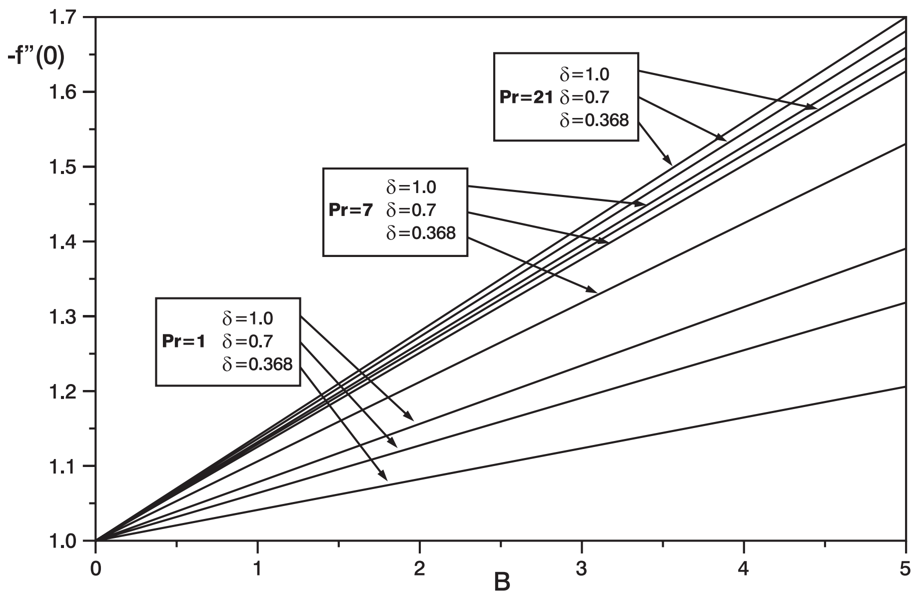

Figure 6. Variation of the wall shear parameter $-f''(0)$.

critical exponent δ corresponds to a specific material of the magnetic particles. So, Fig. 5 shows the variations of $P_2(\eta)$ for three different Prandtl numbers, for different values of δ and for $B=5$. It is observed that for every δ, $P_2(\eta)$ decreases as Pr increases. Similarly, for a specific liquid carrier ($Pr = const.$), $P_2(\eta)$ decreases as δ increases.

The most important flow and heat transfer characteristics are the local skin friction coefficient and the local rate of heat transfer coefficient. These quantities can be defined by the following relations:

$$C_{f_x} = -\frac{2\tau_w}{\rho(cx)^2} \qquad and \qquad Nu_x = \frac{x}{T_c - T_w}\left.\frac{\partial T}{\partial y}\right|_{y=0}, \tag{27}$$

where $\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$ is the wall shear stress, Nu_x is the local Nusselt number and Re_x is the local Reynolds number defined as $Re_x = \rho c x^2/\mu$. Using eqs. (14), (17) and (18) the above mentioned quantities can be written as

$$C_{f_x} = -2f''(0)\,\mathrm{Re}_x^{-1/2} \qquad and \qquad Nu_x = -\left[\Theta_1'(0) + \xi^2\Theta_2'(0)\right]\mathrm{Re}_x^{1/2}, \tag{28}$$

where $f''(0)$ is the dimensionless *wall shear parameter* and $\Theta'(0)$ $\left(= \Theta_1'(0) + \xi^2\Theta_2'(0)\right)$ is the dimensionless *wall heat transfer parameter*. It is apparent that the flow field is affected by the ferromagnetic interaction parameter B. In hydrodynamic case ($B = 0$) P_2 becomes a constant equal to its value zero at infinity (eqs.(21) and (25)) and on the other hand, the flow problem is decoupled from the

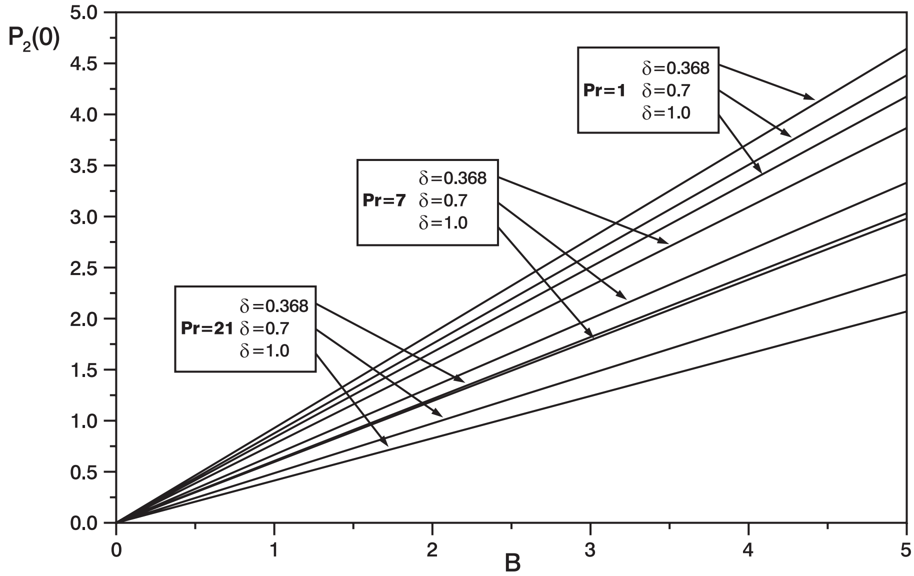

Figure 7. Variation of the wall pressure parameter $P_2(0)$.

thermal energy problem. So, it is more interesting and convenient to replace the dimensionless wall heat transfer parameter $-\Theta'(0) = -\left[\Theta_1'(0)+\xi^2\Theta_2'(0)\right]$ by the dimensionless and independent of the distance ξ ratio $\Theta^*(0) = \frac{\Theta_1'(0)}{\Theta_1'(0)\big|_{B=0}}$ called coefficient of the *heat transfer rate at the wall (sheet)*. Also, the quantity $P_2(0)$ can be defined as the *wall pressure parameter*.

The variations of these three parameters (coefficients), namely $-f''(0)$, $P_2(0)$ and $\Theta^*(0)$, with the ferromagnetic interaction parameter B , are shown in Figs. 6, 7 and 8, respectively, for $Pr = 1, 7, 21$ and for $\delta = 1, 0.7$ and 0.368. It is observed that the wall shear parameter $-f''(0)$, the wall pressure parameter $P_2(0)$ and the heat transfer rate at the wall $\Theta^*(0)$ are varied almost linearly with the ferromagnetic interaction parameter B and $-f''(0)$ as well as $P_2(0)$ increases as B increases whereas $\Theta^*(0)$ decreases as B increases. For every specific value of B and δ , the wall shear parameter $-f''(0)$ increases as Prandtl number Pr increases. It is worth mentioning here that for $Pr = 1$ or 7, $-f''(0)$ increases as the critical exponent δ increases whereas for $Pr = 21$ (blood), $-f''(0)$ decreases as δ increases and this result may be of fundamental importance in the field of Biomagnetic Fluid Dynamics (BFD).

From Fig. 7 it is observed that for every value of the Prandtl number $Pr = 1, 7$ or 21 and for every value of the critical exponent δ , $P_2(0)$ increases linearly with the ferromagnetic interaction parameter B . However, for every specific value of $B \neq 0$, $P_2(0)$ decreases as δ increases or as Pr increases and this decrement is greater for higher values of Pr . Finally, Fig. 8 shows the variations of the coeffi-

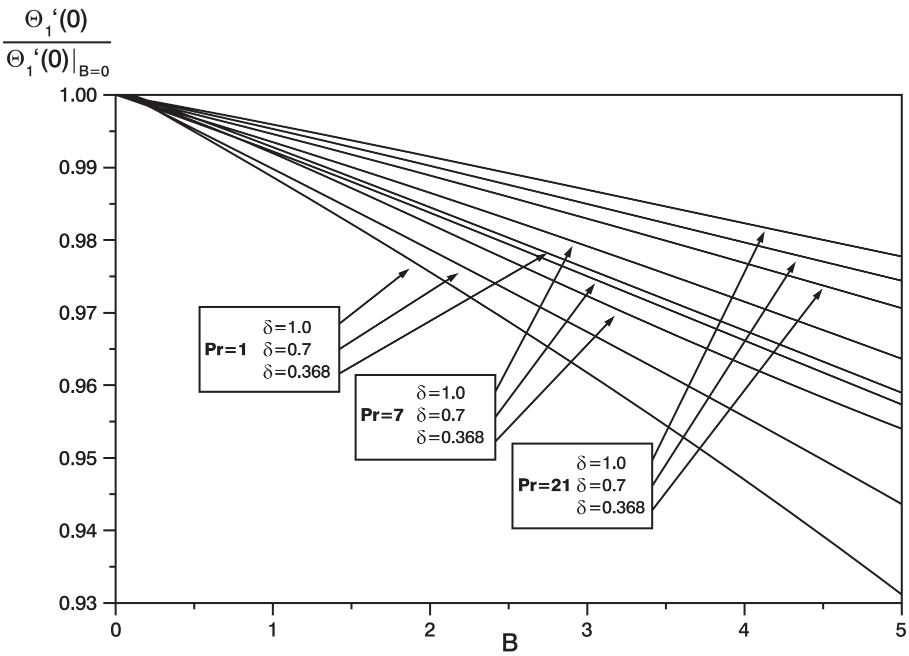

Figure 8. Variation of the wall heat transfer parameter $\Theta^*(0)$.

cient of the heat transfer rate at the wall $\Theta^*(0)$ with B . It is worth emphasizing here that for $Pr = 7$ or 21 , $\Theta^*(0)$ increases as the critical exponent δ increases and the opposite is true for $Pr = 1$. It should be also noticed that the results obtained for $Pr = 1$, 7 and $\delta = 1$ and presented graphically in Figs. 6–8 are identical to those obtained in [2].

5. Concluding remarks

The most important characteristics of the flow field for such type of problems are the dimensionless wall shear parameter $-f''(0)$, the wall pressure parameter $P_2(0)$ and the coefficient of the heat transfer rate at the wall (sheet) $\Theta^*(0)$.

So, we summarize the important results for these quantities as follows:
The variation of the parameters $-f''(0)$, $P_2(0)$ and $\Theta^*(0)$ with the ferromagnetic interaction parameter B is almost linear and $-f''(0)$ as well as $P_2(0)$ increases as B increases whereas $\Theta^*(0)$ decreases as B increases.

The parameters $-f''(0)$ and $\Theta^*(0)$ increase as the Prandtl number Pr increases whereas $P_2(0)$ decreases.

For small values of Pr , such as 1 or 7, the wall shear parameter $-f''(0)$ increases as the critical exponent δ increases. However, for higher values of Pr ,

$-f''(0)$ decreases as δ increases. This decrement of $-f''(0)$, at high Prandtl numbers, can be counterbalanced by the increment, occurred in $-f''(0)$, due to the increase of Prandtl number.

The parameter $P_2(0)$ increases linearly with the ferromagnetic interaction parameter B and decreases as δ increases or as Pr increases. This decrement is greater for higher values of Pr.

For $Pr = 7$ or 21, $\Theta^*(0)$ increases as the critical exponent δ increases whereas the opposite is true for $Pr = 1$.

The results obtained for $Pr = 1$, 7 and $\delta = 1$ are identical to those obtained in [2].

Acknowledgment

This work was supported by the Karatheodoris research program of the University of Patras under project No. 2439.

References

[1] R.E. Rosensweig, *Ferrohydrodynamics*, Cambridge University Press, 1985.
[2] H.I. Andersson and O.A. Valnes, Flow of a heated ferrofluid over a stretching sheet in the presence of a magnetic dipole, *Acta Mechanica*, **128** (1998), 39–47.
[3] J.L. Neuringer and R.E. Rosensweig, Ferrohydrodynamics, *Phys. Fluids* **7** (1964), 1927–1937.
[4] R.L. Bailey, Lesser known applications of ferrofluids, *J. Magnetism Magn. Mater. (MMM)* **39** (1983), 178–182.
[5] R.E. Rosensweig, Magnetic Fluids, *Annu. Rev. Fluid Mech.* **19** (1987), 437–463.
[6] A.C. Eringen and G.A. Maugin, *Electrodynamics of continua II: fluids and complex media*, New York, Springer, 1990.
[7] Y. Haik, J.C. Chen and V.M. Pai, Development of bio-magnetic fluid dynamics, *Proceedings of the IX International Symposium on Transport Properties in Thermal Fluids Engineering, Singapore, Pacific Center of Thermal Fluid Engineering*, S.H. Winoto, Y.T. Chew, N.E. Wijeysundera, eds, Hawaii, U.S.A., June 25–28, 1996, 121–126.
[8] J.L. Neuringer, Some viscous flows of a saturated ferrofluid under the combined influence of thermal and magnetic field gradients. *Int. J. Non-Linear Mech.* **1** (1966), 123–137.
[9] L.J. Crane,Flow past a stretching plate. *J. Appl. Math. Phys.(ZAMP)* **21** (1970), 645–647.
[10] B. Siddappa and B.S Khapate,Rivlin-Ericksen fluid flow past a stretching plate. *Rev. Roum. Sci. Techn.- Me′c . Appl.* **21** (1976), 497–505.
[11] T.C. Chiam,Micropolar fluid flow over a stretching sheet. *J. Appl. Math. Phys.(ZAMP)* **62** (1982), 565–568.
[12] H.I. Andersson and B.S. Dandapat, Flow of a power–law fluid over a stretching sheet. *Stability Appl. Anal. Cont. Media* **1** (1991), 339–347.
[13] K.B. Pavlov, Magnetohydrodynamic flow of an incompressible viscous fluid caused by deformation of a plane surface. *Magnitnaya Gidrodinamika (USSR)* **4** (1974), 146–147.
[14] H.I. Andersson, MHD flow of a viscoelastic fluid past a stretching surface. *Acta Mech.* **95** (1992), 227–230.
[15] H.I. Andersson, K.H. Bech and B.S. Dandapat, Magnetohydrodynamic flow of a power–law fluid over a stretching sheet. *Int. J. Non-Linear Mech.* **27** (1992), 929–936.

[16] A.S. Arrot, B. Heinrich and T.L. Templeton, Phenomenology of Ferromagnetism: I. Effects of Magnetostatics on Susceptibility. *IEEE Trans. on Magnetics* **25** No. 6 (1989), 4364–4373.
[17] N.G. Kafoussias and E.W. Williams, An improved approximation technique to obtain numerical solution of a class of two-point boundary value similarity problems in fluid mechanics. *Int. J. Num. Methods in Fluids* **17** (1993), 145–162.
[18] J.W. Valvano, S. Nho and G.T. Anderson, Analysis of the Weinbaum-Jiji model of blood flow in the Canine Kidney cortex for self-heated thermistors. *J. Biomechanical Engineering* **116** No. 2 (1994), 201.
[19] J.C. Chato, Heat tranfer to blood vessels. *J. Biomechanical Engineering* **102** (1980), 110–118.

E. E. Tzirtzilakis and N. G. Kafoussias
Section of Applied Analysis
Department of Mathematics
University of Patras
26500 Patras
Greece
e–mail: nikaf@math.upatras.gr

(Received: July 19, 2001; revised: January 18, 2002)