A mathematical model for blood flow in magnetic field

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(Received 4 November 2004; accepted 31 May 2005; published online 13 July 2005)

In the present study a mathematical model of biomagnetic fluid dynamics (BFD), suitable for the description of the Newtonian blood flow under the action of an applied magnetic field, is proposed. This model is consistent with the principles of ferrohydrodynamics and magnetohydrodynamics and takes into account both magnetization and electrical conductivity of blood. As a representative application the laminar, incompressible, three-dimensional, fully developed viscous flow of a Newtonian biomagnetic fluid (blood) in a straight rectangular duct is numerically studied under the action of a uniform or a spatially varying magnetic field. The numerical results are obtained using a finite differences numerical technique based on a pressure-linked pseudotransient method on a collocated grid. The flow is appreciably influenced by the application of the magnetic field and in particularly by the strength and the magnetic field gradient. A comparison of the derived results is also made with those obtained using the existing BFD model indicating the necessity of taking into account the electrical conductivity of blood. © *2005 American Institute of Physics*.
[DOI: 10.1063/1.1978807]

I. INTRODUCTION

Biomagnetic fluid dynamics (BFD) is a relatively new area in fluid mechanics investigating the fluid dynamics of biological fluids in the presence of magnetic field. During the last decades extensive research work has been done on the fluid dynamics of biological fluids in the presence of a magnetic field. Numerous applications have been proposed in bioengineering and medical sciences. Among them is the development of magnetic devices for cell separation, targeted transport of drugs using magnetic particles as drug carriers, magnetic wound or cancer tumor treatment causing magnetic hyperthermia, reduction of bleeding during surgeries or provocation of occlusion of the feeding vessels of cancer tumors and development of magnetic tracers.[1–8]

A biomagnetic fluid is a fluid that exists in a living creature and its flow is influenced by the presence of a magnetic field. The most characteristic biomagnetic fluid is blood, which behaves as a magnetic fluid, due to the complex interaction of the intercellular protein, cell membrane and the hemoglobin, a form of iron oxides, which is present at a uniquely high concentration in the mature red blood cells, while its magnetic property is affected by factors such as the state of oxygenation.[9] It is found that the erythrocytes orient with their disk plane parallel to the magnetic field[9–13] and also that blood possesses the property of diamagnetic material when oxygenated and paramagnetic when deoxygenated.[14] Measurements have also been performed for the estimation of the magnetic susceptibility of blood which was found to be 3.5×10^{-6} and -6.6×10^{-7} for the venous and arterial blood, respectively.[15,16] Experiments have been made using a relatively weak magnetic field (1.8 Tesla) and low temperatures (75–295 K).[17] Strong magnetic fields (8 Tesla) were also used on a living rat and the consequence was the reduction of the blood flow and the temperature of the rat.[18] Also experiments have shown that for a magnetic field of the same strength (8 Tesla), the flow rate of human blood in a tube was reduced by 30%.[19]

In order to investigate the flow of a biomagnetic fluid, the BFD model has been developed by Haik *et al.* The mathematical formulation of BFD is analogous to the one of ferrohydrodynamics (FHD), which deals with no induced electric current, and considers that the flow is affected by the magnetization of the fluid in the magnetic field. Thus, the resulting equations of BFD take into account the magnetization of the fluid, as opposed to the formulation of the well-known magnetohydrodynamics (MHD), which deals with conducting fluids, and the corresponding mathematical model ignores the effect of polarization and magnetization. The arising force due to magnetization depends on the existence of a spatially varying magnetic field and in a uniform magnetic field this force vanishes.[1,15,19–21]

Thus, according to the existing BFD model of Haik *et al.*, biofluids are considered electrically poor conductors and the flow laminar, Newtonian and affected only by the magnetization of the fluid in a spatially varying magnetic field. However, blood, in particular, exhibits considerably high static electrical conductivity which is hematocrit and temperature dependent. Over the above, the electrical conductivity of blood varies as the flow rate varies.[22–24]

In the present study the existing mathematical model of BFD,[1,15,19–21] is extended. According to the presented model, both magnetization of BFD, which is consistent with the principles of FHD,[25–31] and the Lorenz force, due to the induced electric current of MHD,[33–35] are taken into account. Various equations describing the magnetization according to FHD are also given.

As a representative application the laminar, incompressible, three-dimensional, fully developed viscous flow of a

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Newtonian biomagnetic fluid (blood) in a straight rectangular duct is numerically studied. Two cases of the applied magnetic field are taken into account, uniform and spatially varying. For the spatially varying magnetic field calculations with different magnetic field gradients are also performed. The system of the partial differential equations, resulting after the introduction of appropriate nondimensional variables, is solved applying an efficient finite differences numerical technique based on a pressure-linked pseudotransient method on a collocated grid.

The same physical problem using the above-mentioned numerical method and for a specific magnetic field gradient has been studied by adopting the existing model of BFD of Haik *et al.* in Ref. 36. The biomagnetic fluid flow in a curved duct adopting the same model of Haik *et al.* and by the use of the SIMPLE method has also been studied in Ref. 37.

The obtained results, for different values of the parameters entering into the problem under consideration, show that the flow is considerably influenced by the presence of the magnetic field. In the presence of spatially varying, as well as uniform magnetic field, the axial velocity component considerably reduces. Especially in the case of spatially varying magnetic field secondary flow arises. It is also obtained that the dominant factor for the formation of the flow field is the form of the magnetic field gradient. A comparison of the obtained results is also made with the existing model of BFD and it is apparent that the inclusion of the Lorentz force of MHD is a necessity for constant and relatively smooth magnetic field gradients. These results, of the effect of an applied magnetic field on the flow of a biomagnetic fluid, encourage further studies for possible useful medical and engineering applications.

II. MATHEMATICAL MODEL

The laminar incompressible flow of a homogeneous Newtonian and electrically conducting biofluid is considered. As the blood flows under the influence of a magnetic field, two major forces will act upon it. The first one is the magnetization force due to the tendency of the erythrocytes to orient with the magnetic field and the second one is the Lorentz force which arises due to the electric current generating from the moving ions in the plasma.

According to the existing mathematical formulation of BFD,[1,15,19–21] which is consistent with the principles of FHD,[25–31] the biofluid is subject to equilibrium magnetization and the apparent viscosity due to the application of the magnetic field is considered negligible. The validation of the assumption of equilibrium magnetization has been proven in Ref. 15. Consequently, the existing model of BFD as well as the model presented in this paper, are both valid for the flow like the one in large blood vessels, where the blood can be considered as a homogeneous and Newtonian fluid.[32] The contribution of the Lorentz force can be incorporated in the mathematical model adopting the principles of MHD.[33–35]

According to the above mentioned considerations the governing equations of flow for an incompressible, homogeneous, Newtonian biofluid are

Continuity equation:

$$\nabla \cdot \mathbf{V} = 0; \tag{1}$$

Momentum equations:

$$\rho \frac{D\mathbf{V}}{Dt} = -\nabla p + \rho \mathbf{F} + \mu \nabla^2 \mathbf{V} + \mathbf{J} \times \mathbf{B} + \mu_0 M \nabla H; \tag{2}$$

Magnetic field equations:

$$\begin{gathered} \nabla \times \mathbf{H} = \mathbf{J} = \sigma(\mathbf{V} \times \mathbf{B}), \\ \nabla \cdot \mathbf{B} = \nabla \cdot (\mathbf{H} + \mathbf{M}) = 0; \end{gathered} \tag{3}$$

Energy equation.

In the nonisothermal case, it is also necessary to consider in the mathematical model, the energy equation containing the temperature T of the fluid. This equation can be written as

$$\rho C_p \frac{DT}{Dt} + \mu_0 T \frac{\partial M}{\partial T} \frac{DH}{Dt} - \frac{\mathbf{J} \cdot \mathbf{J}}{\sigma} = k \nabla^2 T + \mu \Phi. \tag{4}$$

In the above equations $V=(u,v,w)$ is the velocity field, $D/Dt = \partial/\partial t + \mathbf{V}\nabla$ is the Stokes tensor, $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$ with x,y,z the axis of a three-dimensional system, ρ is the fluid density, p is the pressure, $\mathbf{F}$ is the body force per unit volume, μ is the dynamical viscosity, μ_0 is the magnetic permeability of vacuum, $\mathbf{M}$ is the magnetization, $\mathbf{H}$ is the magnetic field intensity, $\mathbf{B}$ is the magnetic induction, σ is the electrical conductivity of the fluid, $\mathbf{J}$ is the density of the electric current, T is the temperature, k is the coefficient of thermal conductivity of the fluid, C_p is the specific heat at constant pressure and Φ is the dissipation function which for the three-dimensional case has the form

$$\Phi = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2.$$

The magnetic force due to magnetization per unit volume is generally $\mu_0(\mathbf{M}\cdot\nabla)\mathbf{H}$.[25–31] Under the assumption of equilibrium magnetization ($\mathbf{M}$ and $\mathbf{H}$ parallel) and keeping in mind that the used magnetic field is solenoid $(\nabla \mathbf{B}=0)$, $\mathbf{J}=\nabla\times\mathbf{H}$ is the induced current and $\mathbf{B}=\mu_0\mathbf{H}$, it is valid, as mentioned in Ref. 30, that

$$\begin{aligned} \mu_0(\mathbf{M}\cdot\nabla)\mathbf{H} &= \mu_0 \frac{M}{H}(\mathbf{H}\cdot\nabla)\mathbf{H} \\ &= \mu_0 \frac{M}{H}\left(\frac{1}{2}\nabla(\mathbf{H}\cdot\mathbf{H}) - \mathbf{H}\times(\nabla\times\mathbf{H})\right) \\ &= \mu_0 \frac{M}{H}\frac{1}{2}\nabla \mathbf{H}^2 - \mu_0 \frac{M}{H}(\mathbf{H}\times\mathbf{J}) \\ &= \mu_0 M \nabla H - \frac{M}{H}(\mathbf{B}\times\mathbf{J}) \\ &= \mu_0 M \nabla H + \frac{M}{H}(\mathbf{J}\times\mathbf{B}). \end{aligned}$$

Thus, the magnetization force along with the Lorenz force (superposition) is

$$\mu_0 M \nabla H + \frac{M}{H}(\mathbf{J}\times\mathbf{B}) + \mathbf{J}\times\mathbf{B} = \mu_0 M \nabla H + \left(1+\frac{M}{H}\right)(\mathbf{J}\times\mathbf{B}).$$

It also holds, as mentioned, for example, in Ref. 30, that $H \gg M$ thus $M/H \ll 1$ and the force due to the imposed magnetic field is finally $\mu_0 M \nabla H + (\mathbf{J}\times\mathbf{B})$.

Thus, the term $\mu_0 M \nabla H$ of Eq. (2), represents the component of the magnetic force, per unit volume and depends on the existence of the magnetic gradient, whereas the term $\mathbf{J}\times\mathbf{B}$ appearing in (2), represents the Lorentz force per unit volume and arises due to the electrical conductivity. These two terms are generally of the same order of magnitude and in MHD the first one is discarded, whereas the second one is discarded in FHD.[25–35]

The term $\mu_0 T(\partial M/\partial T)(DH/Dt)$ of Eq. (4), represents the thermal power per unit volume due to the magnetocaloric effect. This term arises due to the FHD,[25–31] whereas the term $\mathbf{J}\cdot\mathbf{J}/\sigma$ represents the Joule heating and arises due to the MHD.[33–35]

The initial model of BFD of Haik *et al.*[19–21] was developed only for isothermal cases. In their manuscripts they referred that they expected temperature changes of about 2 or 3 °C which they considered negligible. However, this variation of temperature is in reality of great importance for a biological system. The physiological temperature of blood, for example, is about 37 °C. When the temperature of blood rises above 41 °C, irreversible damage occurs in the proteins of plasma and that is the reason that one cannot survive after such high fever.[38]

Moreover, hyperthermia or hypothermia is extensively used for various purposes like cancer tumor treatment, or open heart surgeries.[39,40] Especially for the tumor treatment, the role of the temperature is considerably significant. For increments of 1 °C the time of treatment reduces to the half for a specific biological result like the reduction to one-third of cancer cells of a tumor.[39–41]

Furthermore, hyperthermia caused by the application of magnetic field on injected magnetic fluid has been used for the treatment of eye injuries. By using this method the treatment of a group of patients was possible without using anti-inflammatory medication.[28] Use of hyperthermia, resulting by the application of a magnetic field, has also been reported for the treatment of acid burn necrotic skin wounds of animals. When a magnetic field was applied the temperature raised more than 3 °C and the wounds of 12 and 20 cm^2 closed after 21–26 days, whereas similar wounds, that were not treated magnetically, showed ulcers and scabs, even after 50 days.[42]

Thus, the temperature changes that might occur are important especially for a biofluid like blood and the inclusion of the energy equation is necessary at least at an initial mathematical model of BFD. On the other hand, the adoption of a nonisothermal case leads to the use of the magnetization equation that depends on both temperature and magnetic field strength intensity (see the following section), and the governing equations (1)–(4) are fully coupled. This fact in combination with the computational difficulty investigating such small temperature variations in the flow field may lead to a significantly complicated problem. Thus, depending on the difficulty of a specific physical problem one may initially omit the energy equation, as is also made in the application presented in Sec. III. Although, this is a simplification, it provides a good initial approximation in order to investigate the major effect of the magnetic field in the flow pattern.

A. Magnetization equations

The behavior of a biomagnetic fluid when it is exposed to magnetic field (magnetized) is described by the magnetization property M. Magnetization is the measure of how much the magnetic field affects the magnetic fluid.

In an equilibrium situation the magnetization property is generally determined by the fluid temperature, density and magnetic field intensity. Various equations, describing the dependence of M on these quantities, are given in Refs. 25, 27, 28, 30, 36, 37, and 43–48. The simplest is the linear equation for isothermal cases:[19,27,30,36,37]

$$M = \chi H, \tag{5}$$

where χ is a constant called magnetic susceptibility. The following relation is the linear equation of state, given in Ref. 46:

$$M = K(T_c - T), \tag{6}$$

where K is a constant called pyromagnetic coefficient and T_c is the Curie temperature. Above the Curie temperature the biofluid is not subject to magnetization.

Another equation for magnetization, below the Curie temperature T_c is given in Refs. 44 and 47:

$$M = M_1 \left(\frac{T_c - T}{T_1}\right)^{\beta}, \tag{7}$$

where β is the critical exponent for the spontaneous or saturation magnetization. For iron, $\beta = 0.368$, $M_1 = 54$ Oe and $T_1 = 1.45$ K. An equation involving the magnetic intensity H and the temperature T is given in Refs. 45 and 48:

$$M = K' H (T_c - T). \tag{8}$$

where K' is a constant.

Finally, Higashi *et al.*[9] found that the magnetization process of red blood cells behaves like the following function, known as the Langevin function, which also describes the variation of magnetization for a magnetic fluid:[20,25,27,30]

$$M = mN\left[\coth\left(\frac{\mu_0 m H}{\kappa T}\right) - \frac{\kappa T}{\mu_0 m H}\right], \tag{9}$$

where m is the particle magnetization, N is the number of particles per unit volume and κ the Boltzmann's constant.

The most accurate of the above expressions is (9). However, Eqs. (7) and (8), which have been calculated experimentally in Refs. 47 and 48, respectively, are very good approximations. Moreover, Eq. (5) constitutes a good approximation as experiments show,[15] because especially for

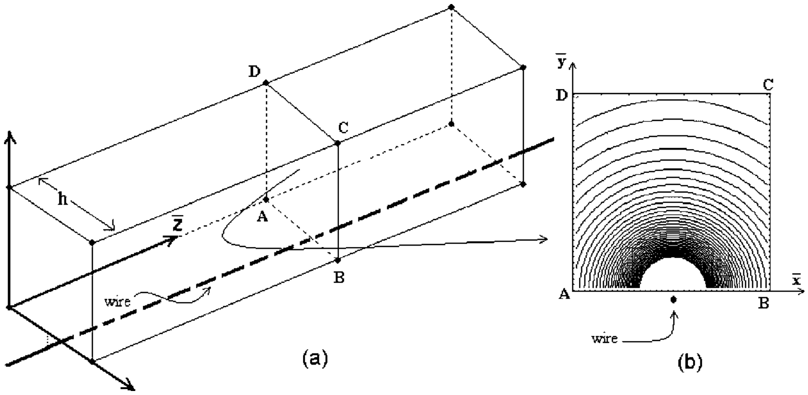

FIG. 1. The flow configuration of case I. The contours of the spatially varying magnetic field of strength H are shown in the ABCD plane of the cross section.

blood, the variations of the temperature in the flow field are not so significant to influence its magnetization. The choice of one of the above-mentioned expressions can be made depending on the physical problem under consideration.

III. APPLICATION

As a simple but representative application the fully developed, laminar, incompressible, three-dimensional, viscous flow, of an electrically conducting biomagnetic fluid (blood), in an impermeable rectangular duct of square cross section of side $\bar{h}$ is considered. Two cases (I and II) concerning the application of the magnetic field are taken into account. In case I a spatially varying magnetic field is generated by an electric current going through a thin wire placed parallel to the axis of symmetry of the bottom plane of the duct, at distance c below it, as shown in Fig. 1(a). The flow is studied at a representative cross section ABCD with the origin of the coordinate system placed at the point A. Consequently, the points A, B, C, and D are A(0,0), B(h,0), C(h,h) and D(0,h) respectively. The wire is placed at the point ($\bar{h}/2,-c$) and thus, the axis of symmetry is $\bar{x}=\bar{h}/2$. The contours of the intensity of the magnetic field H are shown in Fig. 1(b).

In case II a constant magnetic field is applied from below the duct (x-z plane) all over the area where the duct is situated and perpendicularly to the flow (parallel to z-y plane), as shown in Fig. 2(a). The flow is studied in the same with case I representative cross section ABCD with the origin of the coordinate system placed at the point A. The cross section ABCD, where the flow is studied, the way the magnetic field is applied as well as the vectors of the magnetic field induction **B**, are shown in Fig. 2(b). For both cases the temperature changes, for simplification, are considered insignificant and the magnetic field is assumed to be sufficiently strong to saturate the biofluid (equilibrium magnetization), the walls of the duct are assumed electrically conducting.

A. Case I

For the case I, according to the above-mentioned mathematical model, and adopting the simplifications of the hydrodynamic fully developed flow as in Refs. 36, 37, and 50, the dimensional (bar above the quantities) governing equations of the fluid flow, are

$$\frac{\partial \bar{u}}{\partial \bar{x}}+\frac{\partial \bar{v}}{\partial \bar{y}}=0, \quad \text{continuity}, \tag{10}$$

$$\frac{\partial \bar{u}}{\partial \bar{t}}+\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}=-\frac{1}{\bar{\rho}}\frac{\partial \bar{P}}{\partial \bar{x}}+\bar{\nu}\left\{\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}+\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right\}+\frac{\bar{\mu}_0\bar{M}}{\bar{\rho}}\frac{\partial \bar{H}}{\partial \bar{x}}, \quad \bar{x}\text{-momentum}, \tag{11}$$

$$\frac{\partial \bar{v}}{\partial \bar{t}}+\bar{u}\frac{\partial \bar{v}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{v}}{\partial \bar{y}}=-\frac{1}{\bar{\rho}}\frac{\partial \bar{P}}{\partial \bar{y}}+\bar{\nu}\left\{\frac{\partial^2 \bar{v}}{\partial \bar{x}^2}+\frac{\partial^2 \bar{v}}{\partial \bar{y}^2}\right\}+\frac{\bar{\mu}_0\bar{M}}{\bar{\rho}}\frac{\partial \bar{H}}{\partial \bar{y}}, \quad \bar{y}\text{-momentum}, \tag{12}$$

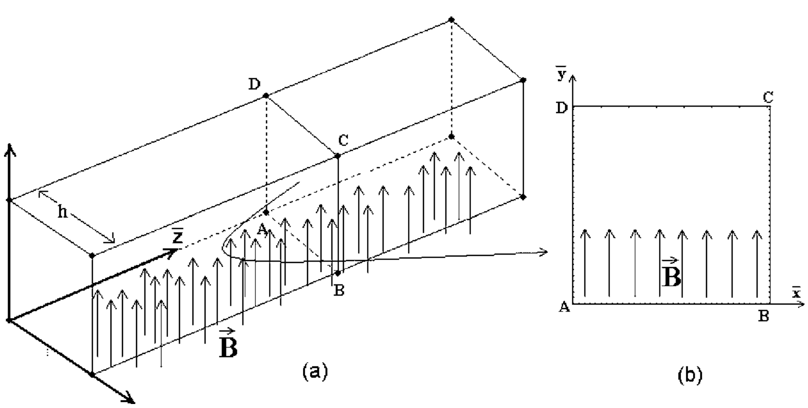

FIG. 2. The flow configuration of case II. The contours of the constant magnetic field of strength H are shown in the ABCD plane of the cross section.

$$\frac{\partial\overline{w}}{\partial\overline{t}}+\overline{u}\frac{\partial\overline{w}}{\partial\overline{x}}+\overline{v}\frac{\partial\overline{w}}{\partial\overline{y}}=-\frac{1}{\overline{\rho}}\frac{\partial\overline{P}}{\partial\overline{z}}+\overline{\nu}\left\{\frac{\partial^2\overline{w}}{\partial\overline{x}^2}+\frac{\partial^2\overline{w}}{\partial\overline{y}^2}\right\}-\frac{\overline{\sigma}_0\overline{B}^2}{\overline{\rho}}\overline{w},\quad \overline{z}\text{-momentum}. \tag{13}$$

The boundary conditions are

$$\overline{y}=0 \quad \text{or}\ \overline{y}=\overline{h} \quad \text{and} \quad 0\leqslant\overline{x}\leqslant\overline{h}: \quad \overline{u}=\overline{v}=\overline{w}=0,$$

$$\overline{x}=0 \quad \text{or}\ \overline{x}=\overline{h} \quad \text{and} \quad 0\leqslant\overline{y}\leqslant\overline{h}: \quad \overline{u}=\overline{v}=\overline{w}=0, \tag{14}$$

where $\overline{B}=\overline{\mu}_0\overline{H}$ is the magnetic field induction and $\overline{\nu}=\overline{\mu}/\overline{\rho}$ is the kinematic viscosity. The variation of the magnetic field is at the $\overline{x}$-$\overline{y}$ plane so the terms, arising due to FHD, $(\overline{\mu}_0\overline{M}/\overline{\rho})\times(\partial\overline{H}/\partial\overline{x})$ and $(\overline{\mu}_0\overline{M}/\overline{\rho})(\partial\overline{H}/\partial\overline{y})$ represent the magnetization force per unit mass in the $\overline{x}$ and $\overline{y}$ directions, respectively.[25–31] In both cases the variation of the magnetic field toward the $\overline{z}$ direction is zero. Thus, the magnetization force is zero at the $\overline{z}$ momentum equation whereas the term $(\overline{\sigma}\overline{B}^2/\overline{\rho})\overline{w}$ arises in the direction of the flow and perpendicularly to the direction of the application of the magnetic field, due to MHD.[33–35] The components of the magnetic field intensity $\overline{H}_x$ and $\overline{H}_y$ along the $\overline{x}$ and $\overline{y}$ coordinates $[H=(\overline{H}_x,\overline{H}_y)]$ are given, respectively, by

$$\overline{H}_x=\frac{\gamma}{2\pi}\frac{\overline{x}-a}{(\overline{x}-a)^2+(\overline{y}-b)^2}$$

$$\overline{H}_y=-\frac{\gamma}{2\pi}\frac{\overline{y}-b}{(\overline{x}-a)^2+(\overline{y}-b)^2},$$

where (a,b) is the point where the magnetic field is applied and γ is the magnetic field strength at this point $(\overline{x}=a,\ \overline{y}=b)$. For the problem under consideration $a=\overline{h}/2$ and $b=-c$ (see Fig. 1).

The magnitude $\overline{H}$, of the magnetic field intensity, is given by

$$\overline{H}(\overline{x},\overline{y})=[\overline{H}_x^2+\overline{H}_y^2]^{1/2}=\frac{\gamma}{2\pi}\frac{1}{\sqrt{(\overline{x}-a)^2+(\overline{y}-b)^2}}. \tag{15}$$

For the variation of magnetization, Eq. (5) for isothermal cases is used:

$$\overline{M}=\overline{\chi}\overline{H}, \tag{16}$$

where $\overline{\chi}$ is the magnetic susceptibility of the biofluid.

B. Case II

For the case II where the magnetic field is uniform, the force due to magnetization does not exist and as a consequence the blood is affected only due to its electrical conductivity. The problem in this case is a trivial problem of MHD and the Lorentz force will arise only at the axial direction in the perpendicular plane of the application of the magnetic field. Thus, the system of the governing equations for this case is

$$\frac{\partial\overline{u}}{\partial\overline{x}}+\frac{\partial\overline{v}}{\partial\overline{y}}=0,\quad \text{continuity}, \tag{17}$$

$$\frac{\partial\overline{u}}{\partial\overline{t}}+\overline{u}\frac{\partial\overline{u}}{\partial\overline{x}}+\overline{v}\frac{\partial\overline{u}}{\partial\overline{y}}=-\frac{1}{\overline{\rho}}\frac{\partial\overline{P}}{\partial\overline{x}}+\overline{\nu}\left\{\frac{\partial^2\overline{u}}{\partial\overline{x}^2}+\frac{\partial^2\overline{u}}{\partial\overline{y}^2}\right\},\quad \overline{x}\text{-momentum}, \tag{18}$$

$$\frac{\partial\overline{v}}{\partial\overline{t}}+\overline{u}\frac{\partial\overline{v}}{\partial\overline{x}}+\overline{v}\frac{\partial\overline{v}}{\partial\overline{y}}=-\frac{1}{\overline{\rho}}\frac{\partial\overline{P}}{\partial\overline{y}}+\overline{\nu}\left\{\frac{\partial^2\overline{v}}{\partial\overline{x}^2}+\frac{\partial^2\overline{v}}{\partial\overline{y}^2}\right\},\quad \overline{y}\text{-momentum}, \tag{19}$$

$$\frac{\partial\overline{w}}{\partial\overline{t}}+\overline{u}\frac{\partial\overline{w}}{\partial\overline{x}}+\overline{v}\frac{\partial\overline{w}}{\partial\overline{y}}=-\frac{1}{\overline{\rho}}\frac{\partial\overline{P}}{\partial\overline{z}}+\overline{\nu}\left\{\frac{\partial^2\overline{w}}{\partial\overline{x}^2}+\frac{\partial^2\overline{w}}{\partial\overline{y}^2}\right\}-\frac{\overline{\sigma}\overline{B}^2}{\overline{\rho}}\overline{w},\quad \overline{z}\text{-momentum}, \tag{20}$$

The boundary conditions remain the same, namely (14), as well as the magnetic field intensity and the variation of magnetization which are given by the relations (15) and (16), respectively.

As far as it could be investigated in the references presented in this paper, the blood in some studies has been treated as an electrically conducting fluid.[49] Thus these studies depended on the principles of MHD and the analogous system of governing equations was that of case II. From this point of view, the polarization (magnetization) of blood in spatially varying magnetic fields had been regarded. The force due to magnetization is generally significant depending also on the magnetic field gradient, as it will be noted below.

According also to the existing model of BFD by Haik *et al.* the blood is treated as an electrically nonconducting fluid.[15,20] Thus, according to this formulation the analogous system of governing equations is that of case I without the corresponding terms to MHD discarding in that way the electrical conductivity of blood. Using the existing BFD model of Haik *et al.*, a uniform magnetic field will have no effect on the flow because the arising force due to magnetization depends on the existence of the magnetic field gradient. Using the model presented in this paper it is possible to study the effect of uniform as well as the effect of spatially varying magnetic field, taking into account all the magnetic properties of blood.

IV. TRANSFORMATION OF EQUATIONS

The following nondimensional variables are introduced:

$$\xi=\frac{\overline{x}}{\overline{h}},\quad \eta=\frac{\overline{y}}{\overline{h}},\quad z=\frac{\overline{z}}{\overline{h}},\quad u=\frac{\overline{u}\overline{h}}{\overline{\nu}},\quad v=\frac{\overline{v}\overline{h}}{\overline{\nu}},\quad w=\frac{\overline{w}\overline{h}}{\overline{\nu}},$$

$$p=\frac{\overline{p}}{\overline{\rho}\overline{\nu}^2/\overline{h}^2},\quad H=\frac{\overline{H}}{H_0},\quad t=\frac{\overline{t}\overline{\mu}}{\overline{\rho}\overline{h}^2}, \tag{21}$$

where $H_0=\overline{H}(\overline{a},0)$. Also the pressure is split as in Refs. 36, 37, and 50:

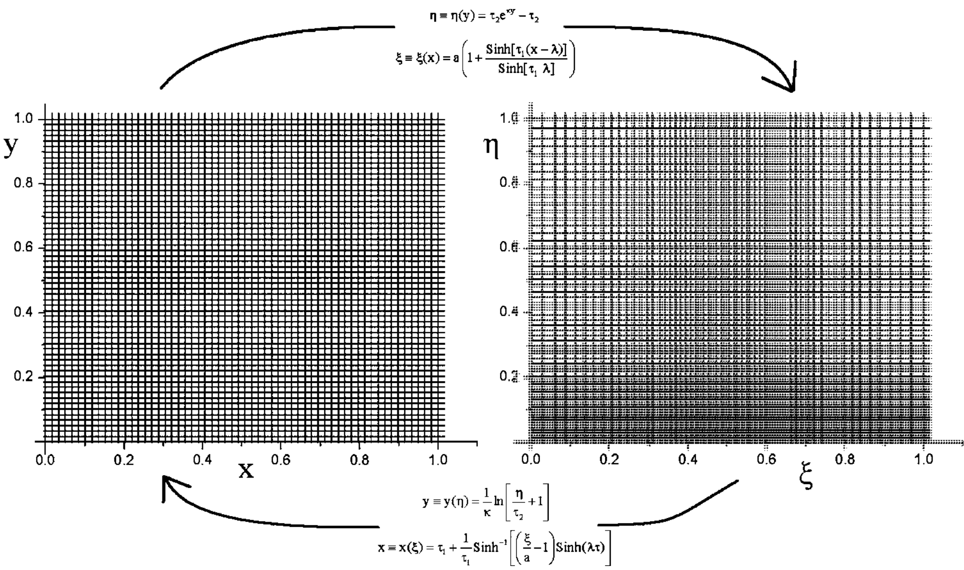

FIG. 3. Grid stretching, the real plane is the ξ-η one, whereas the computational is the x-y plane.

$$P(\xi,\eta,z) = P_1(z) + p(\xi,\eta) \tag{22}$$

for

$$\frac{\partial P}{\partial z} = \frac{\partial P_1}{\partial z} = P_z = \text{const},$$

$$\frac{\partial P}{\partial \xi} = \frac{\partial P}{\partial \xi} = P_\xi, \quad \text{and} \quad \frac{\partial P}{\partial \eta} = \frac{\partial p}{\partial \eta} = P_\eta. \tag{23}$$

A. Case I

By substitution of the relations (15), (16), and (21)–(23) to the governing equations (10)–(13), the following nonlinear system of partial differential equations is obtained:

$$\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta} = 0, \tag{24}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial \xi} + v\frac{\partial u}{\partial \eta} = -P_\xi + \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + Mn_F H\frac{\partial H}{\partial \xi}, \tag{25}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial \xi} + v\frac{\partial v}{\partial \eta} = -P_y + \frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta^2} + Mn_F H\frac{\partial H}{\partial \eta}, \tag{26}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial \xi} + v\frac{\partial w}{\partial \eta} = -P_z + \frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2} - Mn_M H^2 w \tag{27}$$

under the boundary conditions

$$\eta = 0 \quad \text{or} \quad \eta = 1 \quad \text{and} \quad 0 \leqslant \xi \leqslant 1: \quad u = v = w = 0, \tag{28}$$

$$\xi = 0 \quad \text{or} \quad \xi = 1 \quad \text{and} \quad 0 \leqslant \eta \leqslant 1: \quad u = v = w = 0. \tag{29}$$

The two parameters appearing in the problem under consideration are the magnetic numbers Mn_F and Mn_M due to FHD and MHD, respectively,

$$\mathrm{Mn}_F = \frac{\overline{h}^2\overline{\mu}_0\overline{K}\overline{H}_0^2}{\overline{\nu}^2\overline{\rho}} = \frac{\overline{h}^2\overline{B}_0\overline{M}_0}{\overline{\nu}^2\overline{\rho}}, \quad \mathrm{Mn}_M = \frac{\overline{h}^2\overline{\sigma}\overline{\mu}_0^2\overline{H}_0^2}{\overline{\nu}\overline{\rho}} = \frac{\overline{h}^2\overline{\sigma}\overline{B}_0^2}{\overline{\nu}\overline{\rho}}, \tag{30}$$

where $\overline{H}_0 = \overline{H}(\overline{a},0)$, $\overline{B}_0 = \overline{B}(\overline{a},0) = \overline{\mu}_0\overline{H}(\overline{a},0)$ and $\overline{M}_0 = \overline{M}(\overline{a},0) = \overline{\chi}\overline{H}(\overline{a},0)$. Especially the Mn_M number is the square of the widely known MHD Hartmann number.[33–35] Increment of the above mentioned numbers, for a specific fluid ($\overline{\nu},\overline{\rho},\overline{M}_0 = \text{const}$) and for a specific flow problem ($\overline{h} = \text{const}$) means increment of the magnetic field strength $\overline{B}_0$. It is noted also that the two magnetic numbers are also dependent on the height of the duct. Consequently, for a specific magnetic field the magnetic numbers can significantly change by changing the geometric characteristics and this dependence could be of significant importance from a biological view point.

The physical problem described by Eqs. (24)–(27) under the boundary conditions (28) and (29) with $\mathrm{Mn}_M = 0$ is the one obtained by the application of the existing BFD model of Haik *et al.*[20] This problem of flow in a straight rectangular duct using the BFD model of Haik *et al.* has also been studied in Refs. 36 and 37.

B. Case II

For the case II where a constant magnetic field is taken into account, the magnetic force due to FHD vanishes. Thus, the system of equations governing the flow for the case II is the same with that of case I setting $\mathrm{Mn}_F = 0$. For the case II, in the same way as before, the governing system takes the form

$$\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta} = 0, \tag{31}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial \xi} + v\frac{\partial u}{\partial \eta} = -P_\xi + \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}, \tag{32}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial \xi}+v\frac{\partial v}{\partial \eta}=-P_y+\frac{\partial^2 v}{\partial \xi^2}+\frac{\partial^2 v}{\partial \eta^2}, \tag{33}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial \xi}+v\frac{\partial w}{\partial \eta}=-P_z+\frac{\partial^2 w}{\partial \xi^2}+\frac{\partial^2 w}{\partial \eta^2}-\mathrm{Mn}_M H^2 w. \tag{34}$$

The boundary conditions remain the same, namely (28) and (29).

C. Grid configuration

For case I, grid stretching is used.[51] The stretching is pictured in Fig. 3, the computational is the x-y plane and the real one is the ξ-η plane. The relation between the coordinates is given below:

$$\begin{aligned}\xi &\equiv \xi(x)=a\left(1+\frac{\sinh[\tau_1(x-\lambda)]}{\sinh[\tau_1\lambda]}\right),\\ \eta &\equiv \eta(y)=\tau_2 e^{\kappa y}-\tau_2,\end{aligned} \tag{35}$$

$$\begin{aligned}x &\equiv x(\xi)=\tau_1+\frac{1}{\tau_1}\sinh^{-1}\left[\left(\frac{\xi}{a}-1\right)\sinh(\lambda\tau)\right],\\ y &\equiv y(\eta)=\frac{1}{\kappa}\ln\left[\frac{\eta}{\tau_2}+1\right],\end{aligned} \tag{36}$$

where

$$\lambda=\frac{1}{2\tau_1}\ln\left|\frac{1+(e^{\tau_1}-1)a}{1+(e^{-\tau_1}-1)a}\right|, \quad \tau_2=\frac{1}{e^{\kappa}-1} \quad \text{and} \quad \tau_1,\kappa$$

are constants that control the stretching at the ξ and η directions, respectively. The transformation of the derivatives is

$$\frac{\partial}{\partial \xi}=\frac{\partial}{\partial x}\frac{\partial x}{\partial \xi}, \quad \frac{\partial^2}{\partial \xi^2}=\frac{\partial^2}{\partial x^2}\left(\frac{\partial x}{\partial \xi}\right)^2+\frac{\partial}{\partial x}\frac{\partial^2 x}{\partial \xi^2}, \tag{37}$$

$$\frac{\partial}{\partial \eta}=\frac{\partial}{\partial x}\frac{\partial x}{\partial \eta}, \quad \frac{\partial^2}{\partial \eta^2}=\frac{\partial^2}{\partial y^2}\left(\frac{\partial y}{\partial \eta}\right)^2+\frac{\partial}{\partial y}\frac{\partial^2 y}{\partial \eta^2}. \tag{38}$$

Thus, the system of equations (24)–(27) by using the relations (35)–(38), is transformed to the following:

$$\frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial \eta}=0, \tag{39}$$

$$\begin{aligned}&\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}\left(u\frac{\partial x}{\partial \xi}-\frac{\partial^2 x}{\partial \xi^2}\right)+\frac{\partial u}{\partial y}\left(v\frac{\partial y}{\partial \eta}-\frac{\partial^2 y}{\partial \eta^2}\right)\\ &\quad=-\frac{\partial x}{\partial \xi}P_x+\frac{\partial^2 u}{\partial x^2}\left(\frac{\partial x}{\partial \xi}\right)^2+\frac{\partial^2 u}{\partial y^2}\left(\frac{\partial y}{\partial \eta}\right)^2+\mathrm{Mn}_F\, H\frac{\partial x}{\partial \xi}\frac{\partial H}{\partial x},\end{aligned} \tag{40}$$

$$\begin{aligned}&\frac{\partial v}{\partial t}+\frac{\partial v}{\partial x}\left(u\frac{\partial x}{\partial \xi}-\frac{\partial^2 x}{\partial \xi^2}\right)+\frac{\partial v}{\partial y}\left(v\frac{\partial y}{\partial \eta}-\frac{\partial^2 y}{\partial \eta^2}\right)\\ &\quad=-\frac{\partial y}{\partial \eta}P_y+\frac{\partial^2 v}{\partial x^2}\left(\frac{\partial x}{\partial \xi}\right)^2+\frac{\partial^2 v}{\partial y^2}\left(\frac{\partial y}{\partial \eta}\right)^2+\mathrm{Mn}_F\, H\frac{\partial y}{\partial \eta}\frac{\partial H}{\partial y},\end{aligned} \tag{41}$$

TABLE I. Magnetic field strength and corresponding magnetic numbers.

B (Tesla)	Mn_F (Deoxygenated)	Mn_F (Oxygenated)	Mn_M
2	6.90×10^5	-1.3×10^5	0.61
4	2.76×10^6	-5.21×10^5	2.46
6	6.21×10^6	-1.17×10^6	5.53
8	1.10×10^7	-2.08×10^6	9.83
10	1.73×10^7	-3.25×10^6	15.36

$$\begin{aligned}&\frac{\partial w}{\partial t}+\frac{\partial w}{\partial x}\left(u\frac{\partial x}{\partial \xi}-\frac{\partial^2 x}{\partial \xi^2}\right)+\frac{\partial w}{\partial y}\left(v\frac{\partial y}{\partial \eta}-\frac{\partial^2 y}{\partial \eta^2}\right)\\ &\quad=-P_z+\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial x}{\partial \xi}\right)^2+\frac{\partial^2 w}{\partial y^2}\left(\frac{\partial y}{\partial \eta}\right)^2-\mathrm{Mn}_M H^2 w,\end{aligned} \tag{42}$$

where

$$\frac{\partial x}{\partial \xi}=\frac{\sinh[\lambda\tau]}{a\tau\sqrt{1+(\xi/a-1)^2\sinh[\lambda\tau]^2}}, \quad \frac{\partial y}{\partial \eta}=\frac{1}{\kappa(\eta+\tau_2)}, \tag{43}$$

$$\begin{aligned}\frac{\partial^2 x}{\partial \xi^2}&=-\frac{(\xi/a-1)\sinh[\lambda\tau]^3}{a^2\tau(1+(\xi/a-1)^2\sinh[\lambda\tau]^2)^{3/2}},\\ \frac{\partial^2 y}{\partial \eta^2}&=-\frac{1}{\kappa(\eta+\tau_2)^2}.\end{aligned} \tag{44}$$

The boundary conditions are

$$y=0 \quad \text{or } y=1 \quad \text{and} \quad 0\leqslant x\leqslant 1: \quad u=v=w=0, \tag{45}$$

$$x=0 \quad \text{or } x=1 \quad \text{and} \quad 0\leqslant y\leqslant 1: \quad u=v=w=0. \tag{46}$$

For case II of constant magnetic field a uniform grid is used. Thus the system of the governing equations is the one of (31)–(34) replacing ξ with x and η with y, under the boundary conditions (45) and (46). This system of equations is equivalent, for uniform grid, with that of case I by setting $\mathrm{Mn}_F=0$.

V. NUMERICAL SOLUTION AND RESULTS

For the study of the steady state of the flow, for both cases under consideration, a pseudotransient method is used where the time t plays the role of an iteration parameter until the steady state is reached. Thus, the system of equations (39)–(42), subject to the boundary conditions (45) and (46), is solved applying a numerical technique based on the pseudotransient pressure linked equation method (PLEM).[36]

The PLEM scheme is used on a collocated orthogonal grid. The advantage is that the complexity of the discretized differential equations using a staggered grid is avoided. In a collocation grid all the variables are determined at the same grid nodes whereas, in a staggered grid each variable is determined at different grid points.[50]

The magnetic susceptibility of blood [$\overline{\rho}=1050\ \mathrm{kgr/m^3}$, $\overline{v}=3.1\times10^{-6}\ \mathrm{m^2\,s^{-1}}$ (Ref. 32)] has already been measured to be -6.6×10^{-7} for the oxygenated and

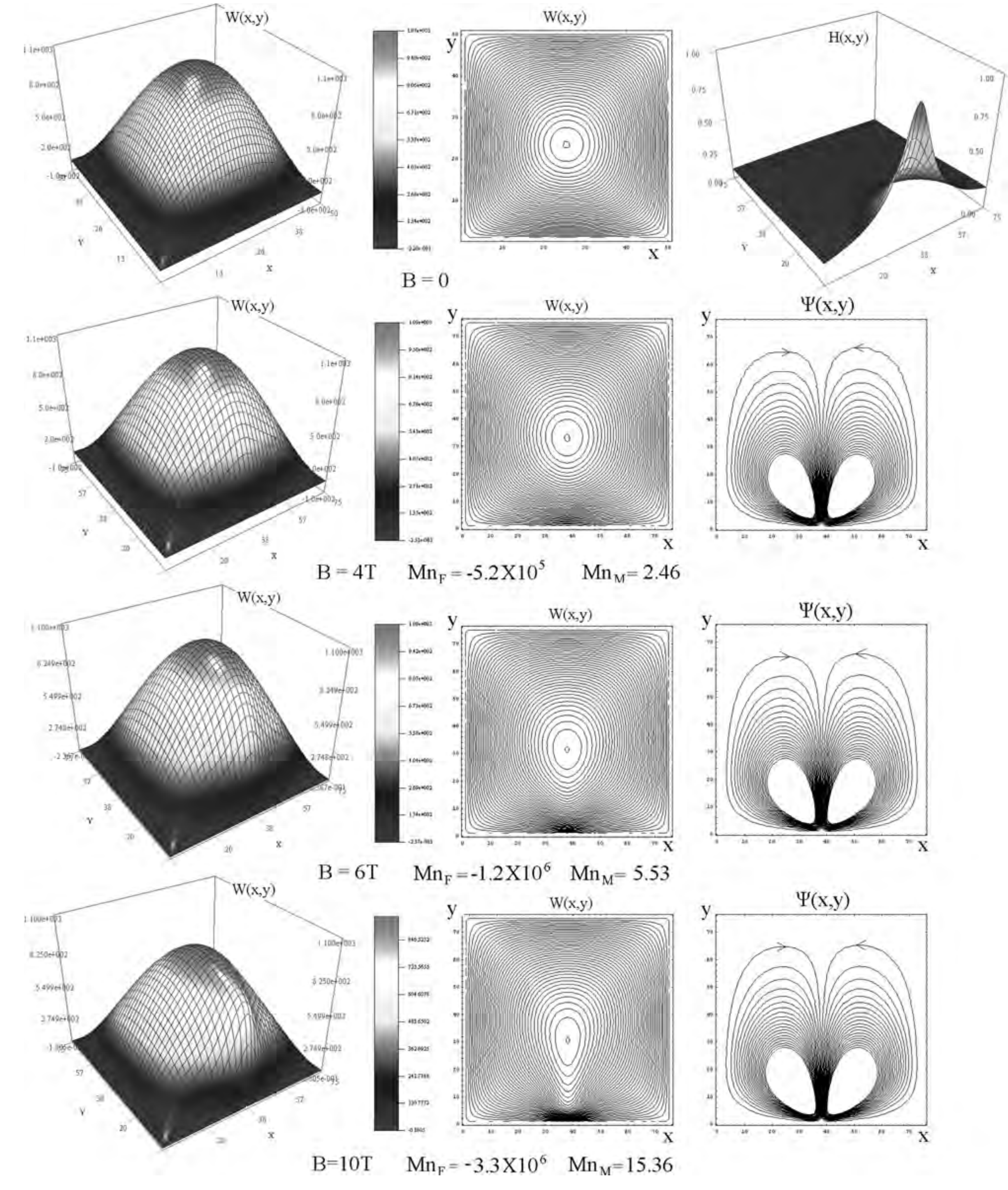

FIG. 4. Axial velocity profiles and corresponding contours for different magnetic field strengths for the oxygenated blood.

3.5×10^{-6} for the deoxygenated blood, respectively.[14–16] Thus the constant $\overline{\chi}$ of the equation of magnetization (16) takes the above-mentioned values depending on which condition of blood is considered. Blood, in particular, also exhibits considerably high static electrical conductivity, which depends on the hematocrit and the temperature. The electrical conductivity $\overline{\sigma}$ of stationary blood was measured to be $0.7\ \mathrm{sm}^{-1}$.[24] The electrical conductivity of flowing blood is always greater than that of the stationary blood. The increment for medium shear rates is about 10% and increases with the increment of the hematocrit.[22] In the current study the electrical conductivity of blood is assumed, for simplicity, temperature independent and equal to $0.8\ \mathrm{sm}^{-1}$.

From the definition of the Mn_F it is apparent that

$$\mathrm{Mn}_F = \frac{\overline{h}^2\overline{\mu_0}\overline{\chi}\overline{H}_0^2}{\overline{\nu}^2\overline{\rho}} = \frac{\overline{h}^2\overline{B}_0^2\overline{\chi}}{\overline{\nu}^2\overline{\rho}\overline{\mu}_0}, \tag{47}$$

where $\overline{\mu}_0$ is the magnetic permeability of vacuum equal to $4\pi\times 10^{-7}$ Hm and $\overline{B}_0$ is the magnetic field induction at the point (0.5,0). For $\overline{h}=2.5\times 10^{-2}$ m, which corresponds to cross-section of a large vessel, it is apparent that $\mathrm{Mn}_F\approx -3\times 10^6$ for the oxygenated and $\mathrm{Mn}_F\approx 1.5\times 10^7$ for the deoxygenated blood, respectively.

From the definition also of Mn_M it is apparent that

$$\mathrm{Mn}_M = \frac{\overline{h}^2\overline{\sigma}\overline{\mu}_0^2\overline{H}_0^2}{\overline{\nu\rho}} = \frac{\overline{h}^2\overline{\sigma}\overline{B}_0^2}{\overline{\nu\rho}}. \tag{48}$$

As it was already mentioned, the blood exhibits magnetization and also holds the property of an electrically conducting fluid. Most of the biofluids due to the existing ions in the body may be influenced by the magnetic field only due to their electrical conductivity. In order to study the effect of the magnetic field due only to the electrical conductivity, Mn_F can be set to zero and Mn_M may vary. If the opposite is done then it is possible to study the effect of the magnetic field due to the magnetization of the fluid. In the present study when the blood is exposed to the uniform magnetic field the Mn_F is set to zero. Thus, studying the blood in uniform magnetic field is qualitatively equivalent with studying one biofluid which exhibits only electrical conductivity. It is also reminded that the existing model of BFD of Haik *et al.* is obtained by the present one by setting $\mathrm{Mn}_M=0$.

The magnetization of blood can be increased by adding artificially created nanoparticles.[21] As a consequence, the Mn_F for a specific magnetic field strength can be increased several orders of magnitude. Equivalently, a specific Mn_F, when the magnetic particles are added, may correspond to much lower (even to some thousand gauss) magnetic field intensity. This practically means that blood except from paramagnetic or diamagnetic material, can be considered as ferromagnetic fluid when the magnetic particles are added.

In the present study, normal blood (without artificial nanoparticles) is considered and the corresponding Mn numbers, for a specific magnetic field of strength intensity B, are calculated from the relations (47) and (48). For both cases I and II the B_0 in the relations (47) and (48) is considered the same with the imposed magnetic field strength $\overline{B}=\overline{\mu}_0\overline{H}$. The difference between the two cases is in the calculation of the dimensionless magnetic field strength intensity $\overline{H}$ in the flow field. For the case I H is calculated by the relations (15) and (21), whereas for the case II H is constant all over the flow domain. Representative values of the magnetic numbers used, for $\overline{h}=2.5\times 10^{-2}$ m, are shown in Table I.

A. Case I

The profiles and corresponding contours for different magnetic field strengths, for varying magnetic field strengths and for the oxygenated blood are shown in Fig. 4. Especially for $B=0$ (hydrodynamic case), the magnetic field strength $H(x,y)$ for the varying case is also pictured. For the other cases of the magnetic field strength, the axial velocity profile $W(x,y)\equiv w$, the contours of this velocity and the stream function $\Psi(x,y)$ in the transverse plane, are pictured in three columns, for each magnetic field strength case. The values printed in the *x*-*y* axes are the number of grid points used for the calculation which in the present case is 75×75. It is obtained that the axial velocity $W(x,y)$ is reduced as the magnetic field strength increases. From the second column it is also clear that the maximum of the axial velocity is somehow "attracted" toward the area where the magnetic source is located. Finally, from the third column it is obtained that a secondary flow is generated in the transverse plane as the magnetic field increases. The secondary flow occurs in the form of two vortices rotating from the outer walls to the center of the duct as shown in the aforementioned Fig. 4. The flow at the transverse plane is obviously symmetric with respect to the point where the magnetic field is applied.

In Fig. 5 the profiles and corresponding contours for different magnetic field strengths for a spatially varying magnetic field and for the deoxygenated blood are pictured. The applied magnetic field intensity $H(x,y)$ is the same with that pictured in Fig. 4 for $B=0$. For this case it is also obtained that the axial velocity $W(x,y)$ is reduced as the magnetic field strength increases. However, in this case the magnetic field seems to "turn away" the blood. The flow at the transverse plane is again symmetric with respect to the point where the magnetic field is applied and it appears in the form of two vortices. In this case these two vortices are rotating in an opposite way with that of the previous case and they are rotating from the center plane toward the two outer walls. From Fig. 5 it is also obtained for the case of deoxygenated blood that the magnetic field affects stronger the axial velocity $W(x,y)$ than that of the oxygenated blood.

The aforementioned results for the deoxygenated blood in spatially varying magnetic field are qualitatively similar to those obtained for a magnetic fluid in varying magnetic field in Refs. 25, 36, and 37. The behavior also of oxygenated blood, which at the transverse plane seems to be "attracted" from the magnetic source, whereas the deoxygenated seems to be "turned away," is justified due to its diamagnetic and paramagnetic nature, respectively.

B. Case II and results in conjunction with case I

The axial velocity profiles for uniform magnetic field are shown in Fig. 6. This case qualitatively corresponds to $\mathrm{Mn}_F=0$ as mentioned above. Under the action of the uniform magnetic field no secondary flow is formed and the magnetic field in that case affects only the axial velocity. As the magnetic field strength increases the axial velocity is reduced.

The mean velocity $\overline{w}_{av}$ passing through the cross section ABCD (see Fig. 1) is given by

$$\overline{w}_{av} = \frac{1}{\overline{h}^2}\int_0^{\overline{h}}\int_0^{\overline{h}} \overline{w}\, d\overline{x}\, d\overline{y}, \tag{49}$$

which, using (21), can be written as

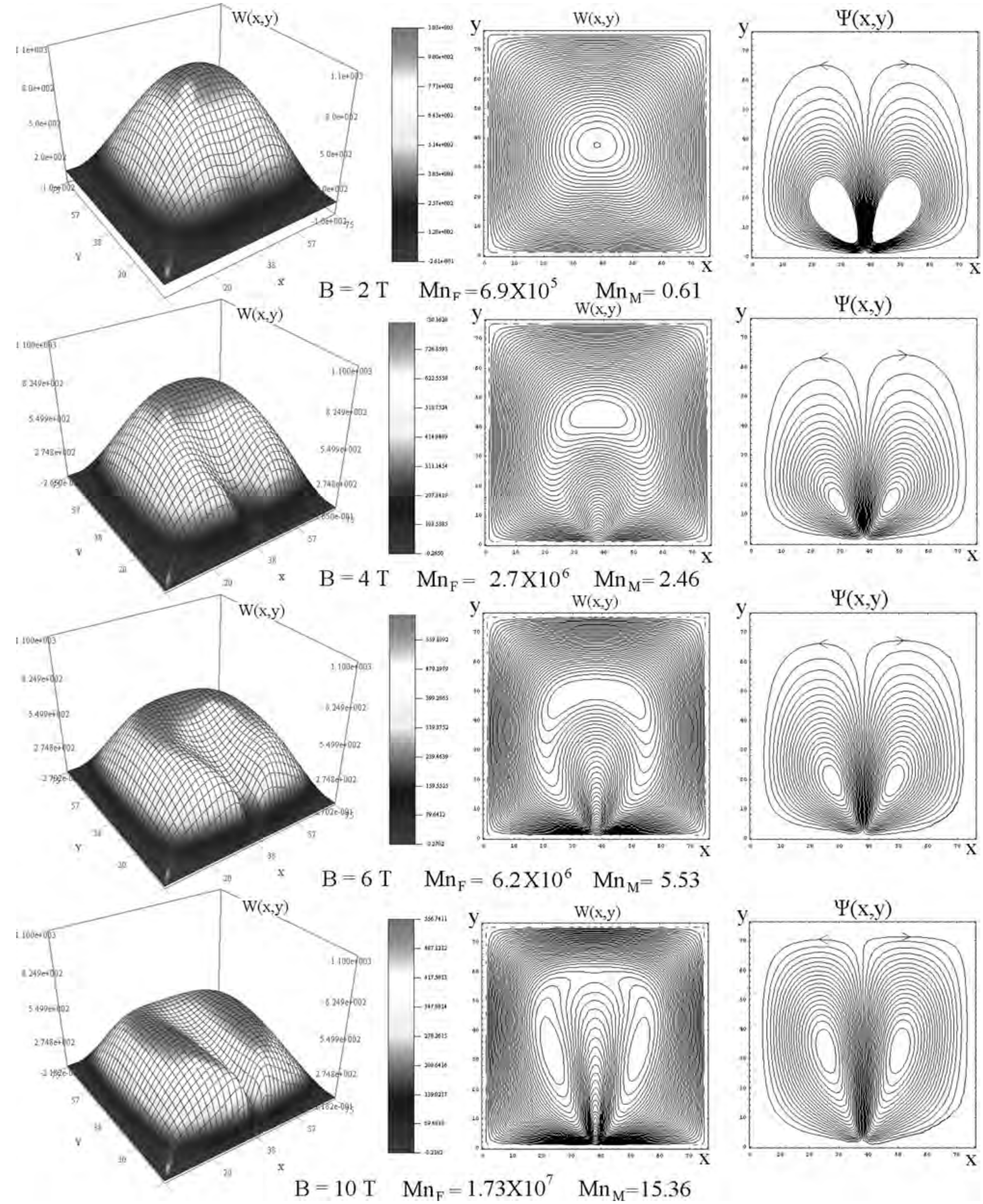

FIG. 5. Axial velocity profiles and corresponding contours for different magnetic field strengths for the deoxygenated blood.

$$\overline{w_{av}} = \frac{\overline{v}}{h}\int_0^1 \int_0^1 w \, dx \, dy. \tag{50}$$

Thus, the Reynolds number defined by $\mathrm{Re}=\overline{w_{av}}\,\overline{h}/\overline{v}$, can be written as

$$\mathrm{Re} = \int_0^1 \int_0^1 w \, dx \, dy \tag{51}$$

and is calculated after the solution of the problem under consideration. From (51) it can be observed that the Re number for the specific problem represents actually the flow rate

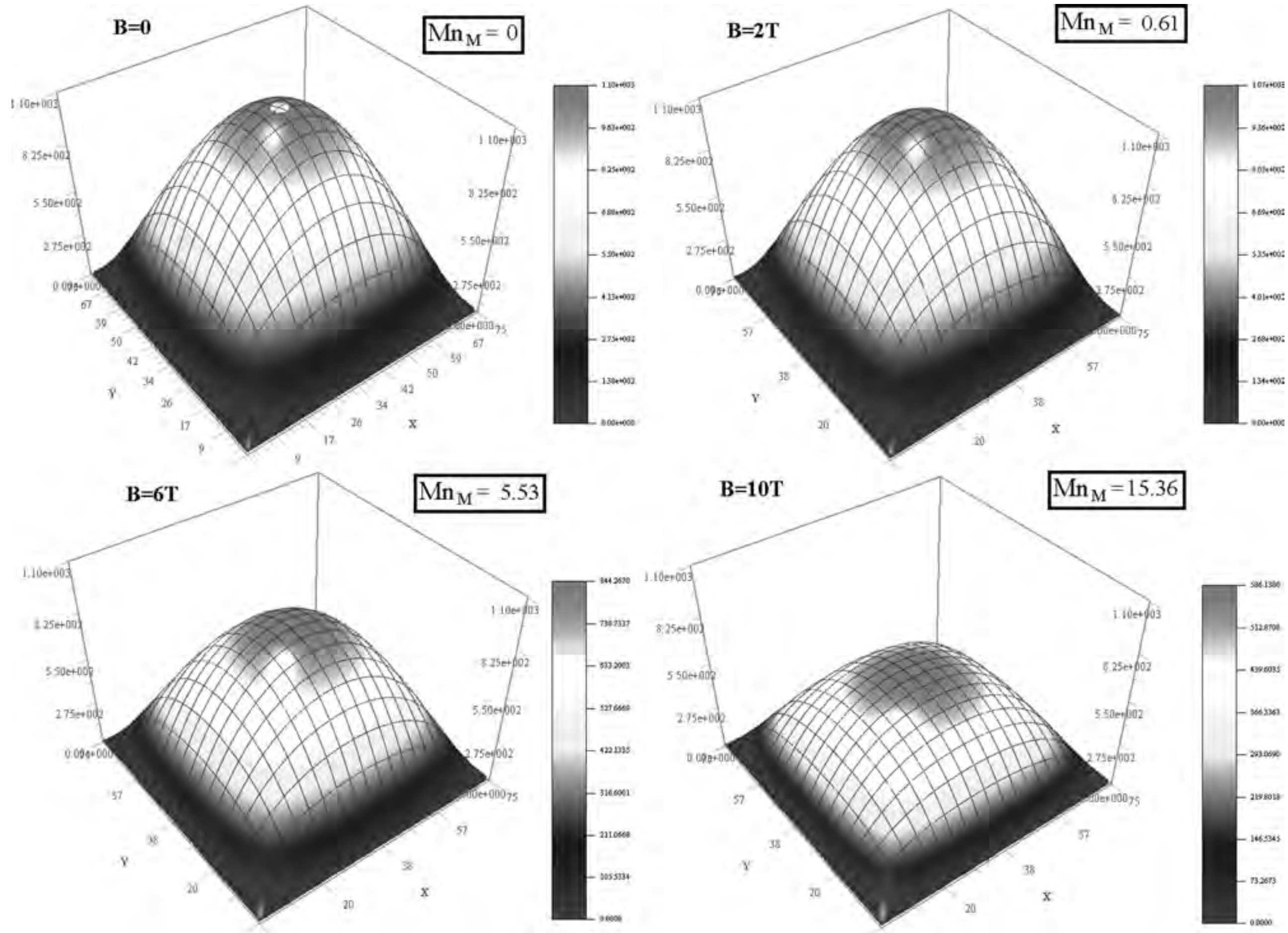

FIG. 6. Axial velocity profiles for different magnetic field strengths for the oxygenated blood and uniform magnetic field.

passing through the cross section ABCD (see Figs. 1 and 2). The number $Re^* = 100[(Re - Re_0)/Re_0]$ is defined in order to investigate the influence of the magnetic field in the flow rate for different values of the magnetic numbers Mn_M and Mn_F, where Re_0 is the Re number for $Mn_M = Mn_F = 0$. Consequently, Re^* represents the percentage change of Re and consequently of flow rate, due to the presence of the magnetic field.

The variation of Re^* with the magnetic field intensity B, is pictured in Fig. 7. It can be observed that the least influence occurs for the varying magnetic field and oxygenated blood. The deoxygenated blood in varying magnetic field is

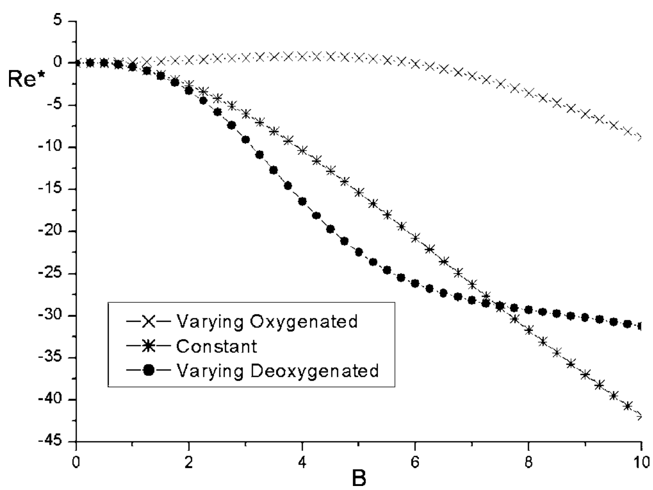

FIG. 7. Percentage variation of Re with the magnetic field strength for constant and varying magnetic field for oxygenated and deoxygenated blood, respectively.

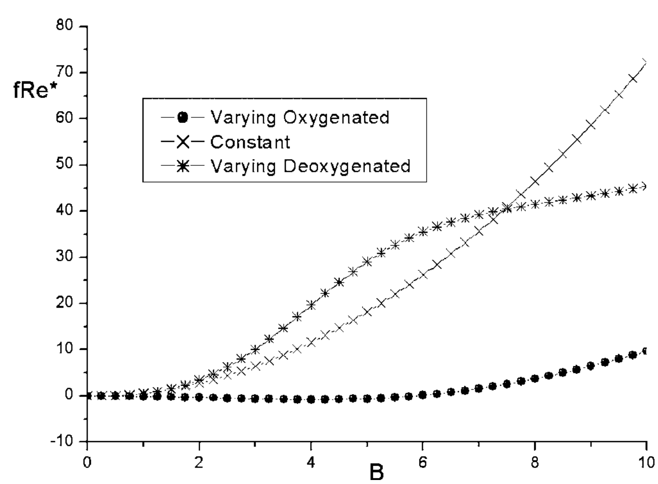

FIG. 8. Percentage variation of f Re with the magnetic field strength for constant and varying magnetic field for oxygenated and deoxygenated blood, respectively.

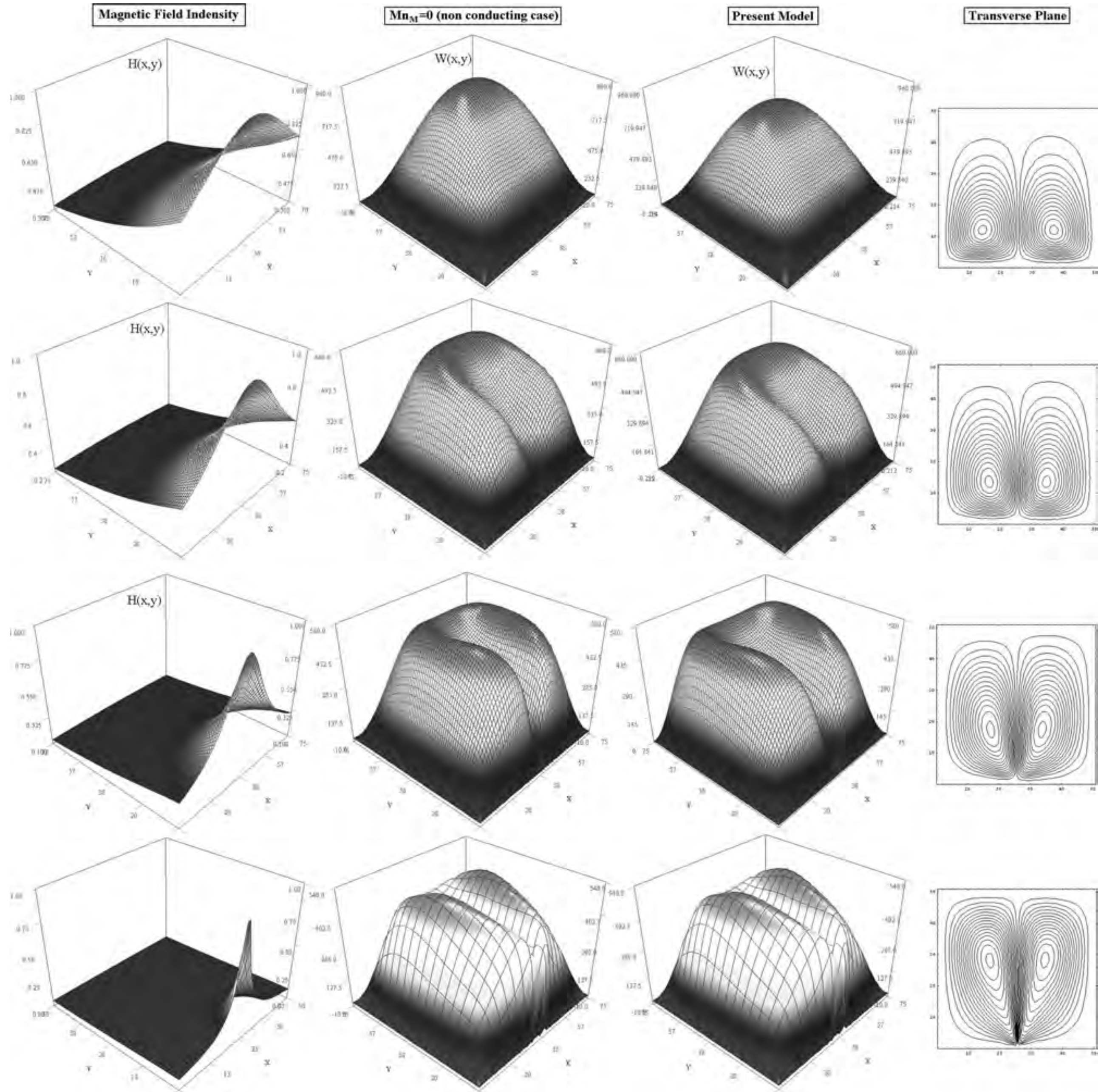

FIG. 9. Axial velocity and contours of the stream function at the transverse plane for deoxygenated blood for $\mathrm{Mn}_M=0$ and for the present model for different forms of magnetic field intensity.

affected more than blood in constant magnetic field for magnetic field strength up to 7.5 Tesla. For greater values of the magnetic field, the blood in constant magnetic field is affected more. For magnetic field of 10 T the flow rate is reduced by 42%, 32% and 10% for constant magnetic field, varying magnetic field and deoxygenated blood and oxygenated blood in varying magnetic field, respectively.

An important flow characteristic is the dimensionless skin friction coefficient f of the flow, given by the expression

$$f = -\frac{\overline{P_z h}}{\overline{\rho w_{av}^2}}. \tag{52}$$

With the use of (21) and (49)–(52) the following nondimensional product is obtained:

$$f\,\mathrm{Re} = -\frac{P_z}{\mathrm{Re}}. \tag{53}$$

In order to investigate the influence of the magnetic field in the blood flow we define, for different values of the magnetic number Mn_F or Mn_M, the number $f\,\mathrm{Re}^* = 100[(f\,\mathrm{Re} - f\,\mathrm{Re}_0)/f\,\mathrm{Re}_0]$, where $f\,\mathrm{Re}_0$ is the $f\,\mathrm{Re}$ number for $\mathrm{Mn}_M = \mathrm{Mn}_F = 0$. Consequently, $f\,\mathrm{Re}^*$ represents the percentage change of $f\,\mathrm{Re}$ due to the presence of the magnetic field. The variation of $f\,\mathrm{Re}^*$ with the magnetic field intensity is shown in Fig. 8. The smallest variation is observed for the varying magnetic field and oxygenated blood where the increment of $f\,\mathrm{Re}$ is 10% for magnetic field 10 T. The highest increment is observed for varying magnetic field and deoxygenated blood

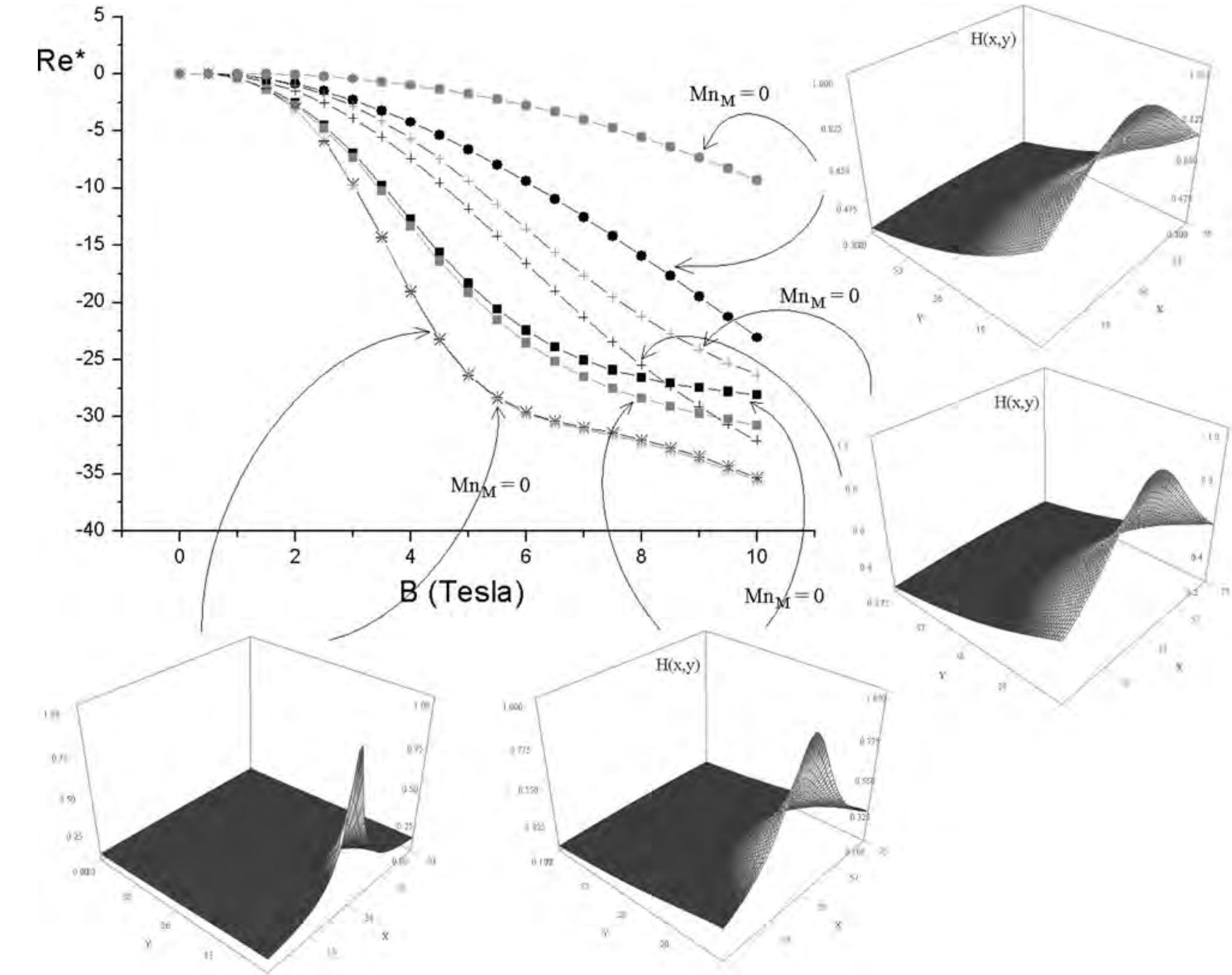

FIG. 10. Percentage variation of $f\,\mathrm{Re}$ with the magnetic field strength for different kids of varying magnetic field for deoxygenated blood for $\mathrm{Mn}_M=0$ and for the present model.

untill 7.5 T where the increment is by 40% and becomes the same with blood in constant magnetic field. For higher magnetic field the $f\,\mathrm{Re}$ increases with the highest rate and at 10 T magnetic field intensity reaches increment of almost 75% whereas the increment for deoxygenated blood is 45%.

It is noted that the percentage variation of $f\,\mathrm{Re}$, $f\,\mathrm{Re}^*$, as well as the percentage variation of Re, Re^*, with the magnetic field intensity are independent of the P_z and thus of the Re number. The results also presented in Figs. 4–6 are qualitatively the same regardless of the P_z number. The presented results are obtained for $P_z=15\,000$, which results to $\mathrm{Re}=526$ for $B=0$. Calculations where also performed for $P_z=5000$ and 25 000.

C. Comparison with the BFD model of Haik *et al.* and effect of the magnetic field gradient

In order to investigate the contribution of the electrical conductivity in the derivation of the mathematical model, calculations where performed setting $\mathrm{Mn}_M=0$ for varying magnetic field [Eqs. (39)–(42)]. For $\mathrm{Mn}_M=0$ the mathematical model is that of BFD proposed by Haik *et al.* in Ref. 20. Figure 9 shows the axial velocity $W(x,y)$ (columns 2 and 3) and contours of the stream function at the transverse plane (4th column) for deoxygenated blood for different forms of magnetic field intensity (1st column) and for $B=10$ T at the magnetic source. It is obtained that, qualitatively speaking, the results are similar and discarding the electrical conductivity ($\mathrm{Mn}_M=0$, 2nd column) leads to overestimation of the axial velocity mostly when the magnetic field gradient is smaller (1st and 2nd row). For sharp magnetic fields tending to zero very fast such as 3rd and 4th row, there is almost no difference. The pattern of the flow field in the transverse plane (4th column) remains the same for each case of magnetic field gradient regardless of the electrical conductivity. Another result that can be obtained from Fig. 9 is the effect of the magnetic field gradient on the flow. The sharper magnetic field leads to stronger secondary flow and greater reduction of the axial velocity and consequently of the flow rate.

The reduction of the flow rate with the magnetic field strength and for different kinds of magnetic field gradients, for deoxygenated blood, is pictured in Fig. 10. For each type of magnetic field gradient the calculations are performed twice. One using the present model and one the existing model of Haik *et al.* ($Mn_M = 0$).

Consideration of electrically nonconducting fluid ($Mn_M = 0$) for the smoothest magnetic field gradient leads to reduction of flow rate almost by 10%, whereas consideration of the electrical conductivity leads to flow reduction almost by 23%. The difference of the estimations between the two models reduces as the magnetic field gradient is sharper and it is getting greater for greater values of the magnetic field intensity. For the sharpest magnetic field there is almost no difference between the two models. Analogous variation is observed for the oxygenated blood.

Summarizing, the electrical conductivity of blood leads to the increase of significant Lorentz force which should be included, together with the magnetization force, in the governing equations of the corresponding mathematical model for blood flow in spatially varying magnetic field. For the case of sharp magnetic field gradients the existing model of BFD of Haik *et al.* could be used as a good approximation whereas for smoother magnetic field gradients the present model is proposed. Finally, for uniform magnetic field, where the model of Haik *et al.* cannot be applied, the blood can be treated as a magnetic fluid in MHD, which is a case that is also covered by the present model.

VI. CONCLUSIONS

In this work a mathematical model of BFD is proposed. As an application the 3D blood flow in a rectangular duct under the action of a magnetic field is numerically studied. Generally, the magnetic field reduces the flow rate. The greater effect of the magnetic field occurs for deoxygenated blood in spatially varying magnetic field and the flow rate can be reduced, for strong magnetic fields, even by 40%. Secondary flow occurs in the transverse plane only for spatially varying magnetic field in the form of two rotating vortices. The form of the magnetic field gradient plays an important role and substantially determines the flow field. The consideration of the electrical conductivity is also necessary especially for the cases of relatively smooth magnetic field gradients and uniform magnetic field. The Lorentz force can be omitted, as it happens according to the model of BFD of Haik *et al.*, only for the regions of application of very sharp magnetic field gradients. For the uniform magnetic field only the axial velocity reduces and no secondary flow occurs. The decrement of flow rate is less than that of the spatially varying magnetic field and reaches the 10% for strong magnetic field.

ACKNOWLEDGMENT

This work was financially supported by the Pythagoras II research program co-funded of the EU program Epeaek II.

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