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# A "discretization" technique for the solution of ODEs ☆

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**Abstract**

A functional-analytic technique was developed in the past for the establishment of unique solutions of ODEs in $H_2(\mathbb{D})$ and $H_1(\mathbb{D})$ and of difference equations in $\ell_2$ and $\ell_1$. This technique is based on two isomorphisms between the involved spaces. In this paper, the two isomorphisms are combined in order to find discrete equivalent counterparts of ODEs, so as to obtain eventually the solution of the ODEs under consideration. As an application, the Duffing equation and the Lorenz system are studied. The results are compared with numerical ones obtained using the 4th order Runge–Kutta method. The advantages of the present method are that, it is accurate, the only errors involved are the round-off errors, it does not depend on the grid used and the obtained solution is proved to be unique.


*Keywords:* Numerical solution; Functional-analytic method; Duffing; Lorenz

## 1. Introduction

In [1,2], a functional-analytic technique appeared for the study of linear ordinary differential equations (ODEs) in the Hardy–Lebesgue Hilbert space of analytic functions defined in $\mathbb{D} =$

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$\{z \in \mathbb{C}: |z| < 1\}$ as

$$H_2(\mathbb{D}) = \left\{ f : \mathbb{D} \to \mathbb{C},\ f(z) = \sum_{n=1}^{\infty} f_n z^{n-1} \text{ with } \sum_{n=1}^{\infty} |f_n|^2 < +\infty \right\}. \tag{1.1}$$

However, it was in [3], that this technique was developed in detail for the study of linear functional–differential equations and systems in $H_2(\mathbb{D})$. This technique was also used in [4–11], for the investigation of analytic and entire solutions of linear ODEs and in [12,13], it was extended to the study of non-linear ODEs, for non-linearities which were powers of the unknown function. In [14,15], the technique was further extended in order to include other kinds of non-linearities. In [12–15] the ODEs are studied in the Banach space

$$H_1(\mathbb{D}) = \left\{ f : \mathbb{D} \to \mathbb{C},\ f(z) = \sum_{n=1}^{\infty} f_n z^{n-1} \text{ with } \sum_{n=1}^{\infty} |f_n| < +\infty \right\}. \tag{1.2}$$

The spaces $H_2(\mathbb{D})$ and $H_1(\mathbb{D})$ are quite useful spaces for studying ODEs since: (a) they contain the polynomial solutions, (b) each element is only one function and not a class of equivalent functions, (c) they contain convergent power-series, (d) they appear naturally in various physical problems (see e.g. [16]).

The main idea of this functional-analytic technique used in [1–15] is the transformation of the ODE under consideration, into an equivalent operator equation in an abstract Hilbert ($H$) or Banach ($H_1$) space, using an isomorphism between $H_2(\mathbb{D})$ and $H$ or $H_1(\mathbb{D})$ and $H_1$.

Meanwhile, in [17], a similar functional-analytic technique was introduced for the study of linear and non-linear ordinary difference equations (O$\Delta$Es) in the Hilbert and Banach space

$$\ell_2 = \left\{ f_n : \mathbb{N} \to \mathbb{R} \text{ with } \sum_{n=1}^{\infty} |f_n|^2 < +\infty \right\}, \tag{1.3}$$

$$\ell_1 = \left\{ f_n : \mathbb{N} \to \mathbb{R} \text{ with } \sum_{n=1}^{\infty} |f_n| < +\infty \right\}, \tag{1.4}$$

respectively. The main idea of this technique, is again the transformation of the O$\Delta$Es under consideration, into an equivalent operator equation in $H$ or $H_1$, using an isomorphism between $\ell_2$ and $H$ or $\ell_1$ and $H_1$. This technique was extended in [18–20] in order to include more general non-linear terms and in [21] in order to study linear systems of O$\Delta$Es.

The spaces $\ell_2$, $\ell_1$ are quite useful spaces for studying O$\Delta$Es, since (a) they give information about the asymptotic behavior of the solution, (b) they are useful for problems of population dynamics, epidemiology and numerical analysis.

The aim of the present paper, is to combine these two functional-analytic techniques in order to find discrete equivalent equations of (systems of) ODEs. At the same time, by studying *one* operator equation in $H_1$, conditions are provided for the existence of a unique, bounded solution in $H_1(\mathbb{D})$ of the (system of) ODEs under consideration and its equivalent discrete counterpart in $\ell_1$. This procedure of finding the discrete equivalent equation of an ODE, which is presented in Section 2, could be considered as an "ideal discretization," since no approximations are made.

In order to illustrate our method we chose two classic examples: the Duffing equation and the Lorenz system, which are two representative models of non-linear dynamics for which there exists an extended bibliography (see e.g. [22,23]) and although they have a "long history," they are studied even nowadays. These two models are studied in Sections 3 and 4 and the obtained

results are compared with numerical results obtained using the commonly used 4th order Runge–Kutta (R–K) method. The numerical results presented in Section 3.3 in the case of the Duffing equation, indicate that our method is better than the R–K method, with respect to the accuracy and the CPU time required.

At this point it worths mentioning the following, regarding the numerical solution of ODEs, which is connected, to some extent, with the present paper: Most of the numerical techniques, especially those involving finite differences, for solving (system of) ODEs are based on their approximation by an appropriate (system of) O$\Delta$Es usually called numerical scheme. One of the most important questions rising from this procedure is if and when the computed solution of the derived numerical scheme converges to the true solution of the initial (system of) ODEs. Another equally important question is whether the computed solution is unique. As far as we could investigate, two major theorems are used in order to assure that a numerically computed solution converges to the true solution of an ODE. The first one is the Lax equivalence theorem and the second one is the Von Neuman stability theorem. The Lax theorem is valid *only for linear* ODEs whereas the Von Neuman theorem is valid *for linear equations with periodic boundary conditions*. For non-linear equations there is no general theorem that assures that a solution computed numerically converges to the true solution of the corresponding non-linear equation. Individual cases presented in the bibliography where the existence and the uniqueness of the computed solution are proved, deal, almost all the times, with simplified cases [24, pp. 92–100], [25, p. 57].

In most applications the existence and uniqueness of the numerically computed solution is generally not studied in a strictly mathematical way. The stability of the numerical method is checked "experimentally" by techniques like changing the number of grid points and comparing the solutions. The reliability of the computed solutions is also investigated by checking their physical meaning and by comparisons with experiments (if any) and other numerical results [25, p. 57]. From a mathematical point of view, when computing solutions of non-linear equations, there is always the danger of instabilities, divergence or, even, estimating the solution of an irrelevant equation than the one the numerical scheme is applied to. Moreover, the loss of accuracy when non-linear terms are involved in the equation is a known phenomenon in numerical analysis.

Thus, the present study constitutes also an effort to calculate the solution of a non-linear (system of) ODEs having assured the existence and uniqueness of the solution by strict mathematical means.

## 2. The method

Denote by $H$ an abstract separable Hilbert space over the real field, with the orthonormal base $\{e_n\}$, $n = 1, 2, 3, \ldots$, and by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the inner product and the norm, respectively. (The field can also be complex, but a restriction to the real field is made, due to the physical problems studied). Define also the shift operator $V$ and its adjoint $V^*$ as:

$$Ve_n = e_{n+1}, \quad n = 1, 2, 3, \ldots, \qquad V^*e_n = e_{n-1}, \quad n = 2, 3, \ldots, \qquad V^*e_1 = 0,$$

the norms of which are $\|V\| = \|V^*\| = 1$, as well as the diagonal operator $C_0$:

$$C_0 e_n = n e_n, \quad n = 1, 2, 3, \ldots.$$

**Proposition 2.1.** [26, p. 3139] *Every point $z \in \mathbb{R}$ with $|z| < 1$, belongs to the point spectrum of $V^*$ and the set of the corresponding eigenelements $f_z = \sum_{n=1}^{\infty} z^{n-1} e_n$, $f_0 = e_1$, forms a complete system in $H$ in the sense that, if $\langle f_z, f \rangle = 0$ for every $z \in \mathbb{R}$, $|z| < 1$, then $f = 0$.*

This proposition allows the establishment of the following 1–1 correspondence between $H$ and $H_2(\mathbb{D})$, $\mathbb{D} = (-1, 1)$ defined by (1.1).

**Proposition 2.2.** [3,26] *The representation*

$$f(z) = \langle f_z, f\rangle = \sum_{n=1}^{\infty} \langle f, e_n\rangle z^{n-1}, \quad |z| < 1 \tag{2.1}$$

*is a one-by-one mapping from $H$ onto $H_2(\mathbb{D})$ which preserves the norm.*

The unique element $f = \sum_{n=1}^{\infty} \langle f, e_n\rangle e_n$ appearing in (2.1) is called the *abstract form* of $f(z)$ in $H$. In general, if $G(f(z))$ is a function from $H_2(\mathbb{D})$ to $H_2(\mathbb{D})$ and $N(f)$ is the unique element in $H$ for which $G(f(z)) = \langle f_z, N(f)\rangle$, then $N(f)$ is called the *abstract form* of $G(f(z))$ in $H$.

Consider now the linear manifold of all $f(z) \in H_2(\mathbb{D})$ which satisfy the condition $\sum_{n=1}^{\infty} |a_n| < +\infty$ and define the norm $\|f(z)\|_{H_1(\mathbb{D})} = \sum_{n=1}^{\infty} |a_n|$. Then this manifold, becomes the Banach space $H_1(\mathbb{D})$ defined by (1.2). Denote also by $H_1$ the corresponding by the representation (2.1), abstract Banach space of the elements $f = \sum_{n=1}^{\infty} \langle f, e_n\rangle e_n \in H$ for which $\sum_{n=1}^{\infty} |\langle f, e_n\rangle| < +\infty$ and by $\|\cdot\|_1$ the norm in $H_1$. (A detailed analysis, why we restrict to $H_1(\mathbb{D})$ and how $H_1(\mathbb{D})$ is connected with $H_2(\mathbb{D})$ is given in [13, p. 385].) For $H_1$ it is known [12, pp. 348–349] that it is invariant under the operators $V^k$, $(V^*)^k$ and under every bounded diagonal operator.

Using representation (2.1) the abstract forms of the terms appearing in the ODEs that we will study were determined in [3,12,13,26]. Indeed:

**Proposition 2.3.** *The following relations hold*:

$$\frac{d^n f(z)}{dz^n} = \big\langle f_z, (C_0 V^*)^n f\big\rangle, \qquad \phi(z) f(z) = \big\langle f_z, \phi(V) f\big\rangle,$$
$$\big[f(z)\big]^n = \big\langle f_z, \big[f_1(V)\big]^{n-1} f\big\rangle,$$

*where $n = 1, 2, \ldots$, $\phi(z) = \sum_{n=1}^{\infty} c_n z^{n-1}$ analytic in a neighborhood of $\bar{\mathbb{D}} = [-1, 1]$, $\phi(V) = \sum_{n=1}^{\infty} c_n V^{n-1}$ and $f_1(V) = \sum_{n=1}^{\infty} \langle f, e_n\rangle V^{n-1}$.*

The next proposition [17] is also fundamental in our approach:

**Proposition 2.4.** *The linear function $\phi : H(H_1) \to \ell_2(\ell_1)$ defined by*

$$\phi(f) = \langle f, e_n\rangle = f(n) = f_n \tag{2.2}$$

*is a* 1–1 *mapping from $H$ ($H_1$) onto $\ell_2$ ($\ell_1$) which preserves the norm, where $\ell_2$, $\ell_1$ are defined by* (1.3) *and* (1.4)*, respectively.*

### *2.1. The "discretization" procedure*

Consider a differential operator $D$ and its corresponding differential equation

$$Df(x) = g(x), \quad |x| < T, \ T > 0. \tag{2.3}$$

**Step 1.** Rewrite (2.3) as

$$\tilde{D} f(z) = g(z), \quad |z| < 1, \quad z = \frac{x}{T}, \tag{2.4}$$

where $\tilde{D}$ a differential operator in $H_1(\mathbb{D})$ and $g(z)$ a known function of $H_1(\mathbb{D})$.

**Step 2.** Use the isomorphism (2.1), i.e. $f(z) = \langle f_z, f \rangle$ and rewrite (2.3) as

$$\big\langle f_z, N(f, g) \big\rangle = 0, \quad \text{where } N \text{ an abstract operator in } H_1. \tag{2.5}$$

**Step 3.** Use the completeness of $\{f_z\}$ and from (2.5) obtain

$$N(f, g) = 0. \tag{2.6}$$

**Step 4.** Take the inner product of both parts of (2.6) with $e_n$ and by using (2.2), i.e. $\langle f, e_n \rangle = f(n)$ derive the difference equation ("numerical scheme")

$$\Delta f_n = g_n, \quad \text{where } \Delta \text{ a difference operator in } \ell_1 \text{ and } g_n \in \ell_1 \text{ known.} \tag{2.7}$$

**Step 5.** Compute $f_n$ from (2.7) and find

$$f(x) = \sum_{n=1}^{\infty} f_n \left( \frac{x}{T} \right)^{n-1}, \quad |x| < T. \tag{2.8}$$

In this way we can find a "numerical scheme" for the ODE under consideration. The associated initial conditions are transformed via (2.1) and (2.2) into equivalent initial conditions for (2.7). From this "numerical scheme," we obtain the coefficients $f_n$ of the truncated solution $f(x) = \sum_{n=1}^{N} f_n (\frac{x}{T})^{n-1}$ of the ODE that we study, where $N$ a finite number.

### *2.2. Choice of T and N*

#### *2.2.1. Choice of T*

Study (2.6) in $H_1$ and obtain conditions, so that it has a unique solution in $H_1$. In order to do this, we follow the ideas of [12–15], [17–20]. The obtained conditions are inequalities involving only $T$, the non-homogeneous term $g(x)$ and the initial conditions and parameters of (2.4). From these conditions obtain a suitable value for $T$. (See Theorems 3.1, 4.1.)

#### *2.2.2. Choice of N*

In order to determine $N$, we take into consideration the fact that $f_n \in \ell_1$. Thus, it is $\lim_{n\to\infty} f_n = 0$. This means that after some $n = k$, the $f_n$ computed in Step 5 will be very small, practically zero (within the round-off error of the computer). Thus $N$ can be chosen greater or equal to $k$.

### *2.3. Advantages of the method*

(1) By studying one operator equation in $H_1$, a unique bounded solution is established for both the (system of) ODEs under consideration and its equivalent (system of) O$\Delta$Es in the spaces $H_1(\mathbb{D})$ and $\ell_1$, respectively.

(2) The method is *accurate*. The only errors encountered in practice are the round-off errors for sufficiently large $N$. On the contrary, the accuracy of the common numerical techniques is

affected by the truncation error. In order to increase the order of the accuracy of a common numerical technique, a new discretization must be employed, which will lead to a new numerical scheme.

(3) The obtained truncated solution (2.8), is the true solution of (2.3), due to the isomorphisms. Also, the presented method *does not depend on the grid used*, since the solution of (2.3) computed by our method, is based on the calculation of the coefficients $f_n$ of (2.8) and is calculated analytically.

(4) The method is very fast as it is shown in Section 3.3.

### *2.4. Disadvantages of the method*

(1) In practice, there are physical problems that admit continuous and not necessarily analytic solutions of (systems of) ODEs. The presented method establishes analytic solutions and there is a possibility of missing an acceptable continuous, but not analytic solution.

(2) The need to use operator techniques in order to implement the method.

## 3. Application to the Duffing equation

### *3.1. Derivation of the discrete equivalent equation.*

In this section we shall apply the presented method to the Duffing equation:

$$x''(t) + \delta x'(t) = ax(t) - \beta\big[x(t)\big]^3 + \gamma\cos(\omega t), \quad |t| < T,\ T > 0, \tag{3.1}$$

$$x(0) = x^0, \qquad x'(0) = x^1, \tag{3.2}$$

where $a, \beta, \gamma, \delta, \omega \in \mathbb{R}$. Equation (3.1) was introduced by Duffing in 1918 in order to describe non-linear mechanical vibrations and, it has been extensively used for the description of forced, damped non-linear oscillations. The term $\delta x'(t)$ represents the damping effect, $ax(t) - \beta[x(t)]^3$ expresses the non-linearity of the oscillations and $\gamma\cos(\omega t)$, stands for the forcing applied to the mechanical system. For $\delta = \beta = 0$ and $a < 0$, (3.1) represents forced simple harmonic motion without damping. For $\delta = \gamma = 0$ (3.1) admits non-linear oscillations in the absence of damping and forcing [22, pp. 214–232, 246–247]. Equation (3.1) has also been proposed in order to describe the vibration of a steel beam attached to an apparatus under the effect of a magnetic field. The first two terms $x''(t)$, $\delta x'(t)$, represent the dissipative effects and the inertia of the beam, respectively, whereas $ax(t) - \beta[x(t)]^3$ represents the non-linear effect of the magnetic field and $\gamma\cos(\omega t)$, represents an external forcing which appears for example from the shaking of the apparatus [23, p. 3].

In order to apply our method, we use the simple transformation $z = \frac{t}{T}$ which restricts (3.1)–(3.2) to the open interval $\mathbb{D}$. In this way (3.1)–(3.2) becomes:

$$x''(z) + \delta T x'(z) = aT^2 x(z) - \beta T^2\big[x(z)\big]^3 + \gamma T^2\cos(\omega T z), \quad |z| < 1, \tag{3.3}$$

$$x(z=0) = x^0, \qquad x'(z=0) = Tx^1. \tag{3.4}$$

According to what mentioned in Section 2 (see Proposition 2.3), Eq. (3.3) can be written, using the abstract forms of the terms involved, as:

$$\begin{aligned}&\big\langle f_z, (C_0V^*)^2x\big\rangle + \delta T\langle f_z, C_0V^*x\rangle = aT^2\langle f_z, x\rangle - \beta T^2\big\langle f_z, \big[x_1(V)\big]^2x\big\rangle + \gamma T^2\langle f_z, g\rangle\\ &\quad\Rightarrow\quad \big\langle f_z, (C_0V^*)^2x + \delta TC_0V^*x - aT^2x + \beta T^2\big[x_1(V)\big]^2x - \gamma T^2g\big\rangle = 0,\end{aligned}$$

where $x_1(V)=\sum_{n=1}^{\infty}\langle x,e_n\rangle V^{n-1}$, $g=\sum_{n=1}^{\infty}a_ne_n$, $a_{2m-1}=\frac{(-1)^{m-1}(\omega T)^{2(m-1)}}{[2(m-1)]!}$, $a_{2m}=0$ or since the set of $f_z$, is complete:

$$(C_0V^*)^2x+\delta TC_0V^*x=aT^2x-\beta T^2\big[x_1(V)\big]^2x+\gamma T^2g. \tag{3.5}$$

If we take the inner product of both parts of (3.5) with $e_n$ we obtain:

$$\begin{aligned}
&\big\langle(C_0V^*)^2x,e_n\big\rangle+\delta T\langle C_0V^*x,e_n\rangle=aT^2\langle x,e_n\rangle-\beta T^2\big\langle\big[x_1(V)\big]^2x,e_n\big\rangle+\gamma T^2\langle g,e_n\rangle\\
&\Rightarrow\quad \langle C_0V^*x,ne_{n+1}\rangle+\delta T\langle x,ne_{n+1}\rangle\\
&\qquad=aT^2\langle x,e_n\rangle-\beta T^2\left\langle\sum_{k=1}^{\infty}\langle x,e_k\rangle V^{k-1}x_1(V)x,e_n\right\rangle+\gamma T^2a_n\\
&\Rightarrow\quad n(n+1)\langle x,e_{n+2}\rangle+n\delta T\langle x,e_{n+1}\rangle\\
&\qquad=aT^2\langle x,e_n\rangle-\beta T^2\sum_{k=1}^{\infty}\langle x,e_k\rangle\left\langle\sum_{s=1}^{\infty}\langle x,e_s\rangle V^{s-1}x,e_{n-k+1}\right\rangle+\gamma T^2a_n\\
&\Rightarrow\quad n(n+1)\langle x,e_{n+2}\rangle+n\delta T\langle x,e_{n+1}\rangle\\
&\qquad=aT^2\langle x,e_n\rangle-\beta T^2\sum_{k=1}^{\infty}\langle x,e_k\rangle\sum_{s=1}^{\infty}\langle x,e_s\rangle\langle x,e_{n-k+1-s+1}\rangle+\gamma T^2a_n\\
&\Rightarrow\quad n(n+1)\langle x,e_{n+2}\rangle+n\delta T\langle x,e_{n+1}\rangle\\
&\qquad=aT^2\langle x,e_n\rangle-\beta T^2\sum_{k=1}^{n}\langle x,e_k\rangle\sum_{s=1}^{n-k+1}\langle x,e_s\rangle\langle x,e_{n-k-s+2}\rangle+\gamma T^2a_n
\end{aligned}\tag{3.6}$$

since the index of $e$ should always be greater or equal to 1. Using (2.2) we find from (3.6) the discrete equivalent equation (or "numerical scheme"):

$$\begin{aligned}
x_{n+2}=&\frac{\gamma T^2}{n(n+1)}a_n-\frac{\delta T}{n+1}x_{n+1}+\frac{aT^2}{n(n+1)}x_n\\
&-\frac{\beta T^2}{n(n+1)}\sum_{k=1}^{n}x_k\sum_{s=1}^{n-k+1}x_sx_{n-k-s+2},\quad \forall n=1,2,\ldots.
\end{aligned}\tag{3.7}$$

From the initial conditions (3.4), we find

$$x(z=0)=x^0\quad\Rightarrow\quad\sum_{n=1}^{\infty}x_nz^{n-1}\Big|_{z=0}=x^0\quad\Rightarrow\quad x_1=x^0, \tag{3.8}$$

$$x'(z=0)=Tx^1\quad\Rightarrow\quad\sum_{n=2}^{\infty}(n-1)x_nz^{n-2}\Big|_{z=0}=Tx^1\quad\Rightarrow\quad x_2=Tx^1. \tag{3.9}$$

From (3.7)–(3.9) we find the coefficients $x_n$ of the solution $x(t)=\sum_{n=1}^{\infty}x_n(\frac{t}{T})^{n-1}$ of (3.1)–(3.2), for $|t|<T$.

### *3.2. Existence and uniqueness conditions*

**Theorem 3.1.** *Let*

$$|\delta|T+|\alpha|T^2<2, \tag{3.10}$$

$$\big|x^0\big|+\big|x^1\big|T+\frac{|\gamma|T^2}{2}\big\|\cos(\omega Tz)\big\|_{H_1(\mathbb{D})}<\frac{(2-|\delta|T-|\alpha|T^2)^{3/2}}{3T\sqrt{|\beta|}}, \tag{3.11}$$

*where $\alpha$, $\beta$, $\gamma$, $\delta$, $\omega$, $x^0$ and $x^1$, are the parameters and the initial conditions of the Duffing equation* (3.1). *Then the operator equation* (3.5) *has a unique solution in $H_1$ bounded by $\sqrt{\frac{2-|\delta|T-|\alpha|T^2}{3|\beta|T^2}}$. Equivalently, the $O\Delta E$* (3.7) *together with the conditions* (3.8)–(3.9) *has a unique solution in $\ell_1$ and the initial value problem* (3.3)–(3.4) *has a unique solution in $H_1(\mathbb{D})$. Moreover, $\sum_{n=1}^{\infty}|x_n| < \sqrt{\frac{2-|\delta|T-|\alpha|T^2}{3|\beta|T^2}}$.*

**Remark 3.2.** The preceding theorem holds also when $z \in \mathbb{C}$. The only difference would be that the appearing absolute values would become moduli.

**Proof.** The operator equation (3.5) can also be written as

$$C_0(C_0+I)(V^*)^2x + \delta T C_0 V^* x - aT^2 x = \gamma T^2 g - \beta T^2\big[x_1(V)\big]^2 x,$$

since $(C_0V^*)^2 = C_0(C_0+I)(V^*)^2$ (see [3, p. 91]) or

$$(V^*)^2x + \delta T B_1 V^* x - aT^2 Bx = \gamma T^2 Bg - \beta T^2 B\big[x_1(V)\big]^2 x, \tag{3.12}$$

where $B$, $B_1$ are the diagonal operators: $Be_n = \frac{1}{n(n+1)}e_n$, $B_1e_n = \frac{1}{n+1}e_n$, $n = 1, 2, \ldots$, with norms $\|B\|_1 = \|B_1\|_1 = \frac{1}{2}$. Since $V^*e_1 = 0$, (3.12) becomes:

$$\begin{aligned} &x + \delta T V^2 B_1 V^* x - aT^2 V^2 Bx \\ &\quad = c_1e_1 + c_2e_2 + \gamma T^2 V^2 Bg - \beta T^2 V^2 B\big[x_1(V)\big]^2 x, \end{aligned} \tag{3.13}$$

where $c_1$, $c_2$ are arbitrary constants. In order to determine $c_1$, $c_2$, we take the inner product of both parts of (3.13) with $e_1$ and $e_2$, respectively, and we find:

$$c_1 = \langle x, e_1\rangle = x_1 = x^0, \qquad c_2 = \langle x, e_2\rangle = x_2 = Tx^1.$$

Thus Eq. (3.13) becomes:

$$\begin{aligned} &\big(I + \delta T V^2 B_1 V^* - aT^2 V^2 B\big)x \\ &\quad = x^0e_1 + Tx^1e_2 + \gamma T^2 V^2 Bg - \beta T^2 V^2 B\big[x_1(V)\big]^2 x. \end{aligned} \tag{3.14}$$

In order to invert the operator $I + \delta T V^2 B_1 V^* - aT^2 V^2 B$ we use the following well-known result of operator theory (see for example [27, pp. 70–71]):

**Theorem 3.3.** *Let $K$ be a linear bounded operator of a Hilbert space $H$ with $\|K\| < 1$. Then $(I-K)^{-1}$ is defined on all $H$ and $\|(I-K)^{-1}\| \leqslant \frac{1}{1-\|K\|}$.*

In our case it is $K = -\delta T V^2 B_1 V^* + aT^2 V^2 B$ and $\|K\|_1 \leqslant \frac{|\delta|T+|a|T^2}{2} < 1$, due to (3.10). Thus $(I + \delta T V^2 B_1 V^* - aT^2 V^2 B)^{-1}$ is uniquely determined in all $H_1$ and bounded by $\|(I + \delta T V^2 B_1 V^* - aT^2 V^2 B)^{-1}\| \leqslant \frac{2}{2-|\delta|T-|\alpha|T^2}$. Then Eq. (3.14) becomes:

$$\begin{aligned} x = &\big(I + \delta T V^2 B_1 V^* - aT^2 V^2 B\big)^{-1}\big[x^0e_1 + Tx^1e_2 \\ &+ \gamma T^2 V^2 Bg - \beta T^2 V^2 B\big[x_1(V)\big]^2 x\big] = \phi(x). \end{aligned} \tag{3.15}$$

Next we use the following fixed point theorem of Earle and Hamilton [28]:

**Theorem 3.4.** *If $p: X \to X$ is holomorphic, i.e. its Fréchet derivative exists, and $p(X)$ lies strictly inside $X$, then $p$ has a unique fixed point in $X$, where $X$ is a bounded, connected and open subset of a Banach space $Y$. (By saying that a subset $X'$ of $X$ lies strictly inside $X$ we mean that there exists an $\epsilon > 0$ such that $\|x' - y\| > \epsilon$ for all $x' \in X'$ and $y \in Y - X$.)*

Let $\|x\|_1 \leqslant R$, $R$ sufficiently large but finite. Then we obtain from (3.15):

$$\begin{aligned} \big\|\phi(x)\big\|_1 &\leqslant \frac{2}{2 - |\delta|T - |\alpha|T^2}\left(\big|x^0\big| + \big|x^1\big|T + \frac{|\gamma|T^2}{2}\|g\|_1\right) + \frac{|\beta|T^2}{2 - |\delta|T - |\alpha|T^2}\|x\|_1^3 \\ \Rightarrow \quad \big\|\phi(x)\big\|_1 &\leqslant \frac{2}{2 - |\delta|T - |\alpha|T^2}\left(\big|x^0\big| + \big|x^1\big|T + \frac{|\gamma|T^2}{2}\|g\|_1\right) \\ &\quad + \frac{|\beta|T^2}{2 - |\delta|T - |\alpha|T^2}R^3, \end{aligned} \tag{3.16}$$

since $\|x_1(V)\|_1 = \|x\|_1$ (see [12, p. 349]). Let

$$P(R) = R - \frac{|\beta|T^2}{2 - |\delta|T - |\alpha|T^2}R^3.$$

This function has the maximum

$$P(R_0) = \frac{2}{3}R_0 \quad \text{at } R_0 = \sqrt{\frac{2 - |\delta|T - |\alpha|T^2}{3|\beta|T^2}}.$$

Then, if

$$\frac{2}{2 - |\delta|T - |\alpha|T^2}\left(\big|x^0\big| + \big|x^1\big|T + \frac{|\gamma|T^2}{2}\|g\|_1\right) \leqslant P(R_0) - \epsilon,$$

where $\epsilon > 0$ arbitrary we have $|\phi(x)| \leqslant R_0 - \epsilon < R_0$ and $\phi$ is a holomorphic map (since $[x_1(V)]^2$ is Fréchet differentiable, see [12, p. 355]) from $S(0, R_0) = \{x \in H_1\colon \|x\|_1 < R_0\}$ strictly inside $S(0, R_0)$. Thus if (3.11) holds, Theorem 3.4 can be applied to (3.15), which completes the proof of the theorem. □

### *3.3. Numerical results*

First we consider the simplified case $\beta = \gamma = 0$, $\alpha = -1$ and $\delta = 0.1$. The exact closed-form solution is

$$x(t) = e^{-\delta t/2}\left[c_1 \cos\left(\frac{\sqrt{-B}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{-B}}{2}t\right)\right],$$

where

$$B = \delta^2 + 4\alpha < 0, \qquad c_1 = x(0), \qquad c_2 = \frac{2}{\sqrt{-B}}\left(x'(0) + \frac{\delta}{2}c_1\right).$$

For the derivation of the numerical results the well-known fourth order Runge–Kutta (R–K) method (of order $(\Delta t)^4$) was used with time step $\Delta t = 10^{-3}$. For this time step, the expected accuracy is 12 significant digits. For the comparison of our method with the R–K we calculate the solution $x_i$ at discrete times $t_i$, $i = 1, \ldots, M$, using both methods. The solution on the above mentioned grid points for the present method is attained using relation (2.8).

A limitation of our method seems to be the upper bound $T$ of the time $t$ where the solution is calculated (see Theorem 3.1). This practically means that we cannot estimate trajectories formed after a long period of time. However, this difficulty can be easily overcome working as follows: We first calculate the solution with initial value $x(0) = x_0$, $x'(0) = x_0^1$ and for a suitable value of $T$ indicated by Theorem 3.1. Thus, the solution can be found until the corresponding to $T$, $t_1$ and will be $x(t_1) = x_1$, $x'(t_1) = x_1^1$. Considering now $x(t_1) = x_1$, $x'(t_1) = x_1^1$ known initial values, we calculate the solution. In that way we solve successive initial value problems considering as initial values of the next problem the last calculated values of $x(t)$ and $x'(t)$ of the previous one. The value of $T$ for all the initial value problems is chosen in such a way that Theorem 3.1 is satisfied. In this case, we used $T = 1.36$ and we calculated trajectories of length 30 time units. This means 30,000 iteration steps for the R–K and $M = 22$ for the present method. Each one of the intervals for $T = 1.36$ was divided in 1360 grid points and thus the solution is calculated by both methods for the same discrete values of time. The number of terms $N$ of the series $x(t) = \sum_{n=1}^{N} x_n (\frac{t}{T})^{n-1}$ calculated by our method is 30. Numerical experiments showed that the solution with the present method as well as the R–K method retains 12–13 significant digits accuracy.

One way to compare and investigate the accuracy of the two methods, is to track the residual of the solution in time. If $\bar{x}$ is an exact solution of the Duffing equation, (3.1) will be satisfied. If the solution $x(t)$ computed either numerically or by the present method, is substituted using computer into (3.1) it will not give exactly zero. Thus the satisfied equation is

$$x''(t) + \delta x'(t) - ax(t) + \beta\big[x(t)\big]^3 - \gamma\cos(\omega t) = R, \tag{3.17}$$

where $R$ is called the residual of the equation. It is expected for a convergent numerical scheme that $R \to 0$ as the time step $\Delta t \to 0$. Let us consider the solution $x$ computed at grid points $t_i$, namely $x(t_i)$, where $i = 1, \ldots, N$ with initial values at $t = 0$: $x(0)$, $x'(0)$. If the solution $x(t_i)$ is substituted to (3.17) it will give an $R_i$ as residual. We define as $R$ the maximum absolute value of $R_i$, i.e. $R = \max|R_i|$, $i = 1, \ldots, N$, and as $\bar{R}$ the mean value of the absolute value of $R_i$, i.e. $\bar{R} = \frac{1}{N}\sum_{i=1}^{N}|R_i|$, $i = 1, \ldots, N$. The $\bar{R}$ can be used in order to observe the overall behavior (over all $t_i$) of a computed solution, whereas the $R$ can be used for a very strict observation and reveals divergence of the solution even in one grid point $t_i$. For the estimation of the $R_i$ it is necessary to evaluate the derivatives appearing in (3.17). In order to accomplish this, we use second order ($O((\Delta t)^2)$) central differences. The representation of the derivatives are

$$x'(t_i) = \frac{x(t_{i+1}) - x(t_{i-1})}{2\Delta t}, \qquad x''(t_i) = \frac{x(t_{i+1}) - 2x(t_i) + x(t_{i-1})}{\Delta t^2}.$$

Due to the estimation of the derivatives the $\bar{R}$ for the R–K method will increase but should be at most of the order of $10^{-6}$ since the time step is $10^{-3}$. The $R$ and $\bar{R}$ for the R–K and the present method, for various initial values and for the studied case are presented in Table 1. As it is expected the two methods are equivalent with respect to the criterion of $R$ and $\bar{R}$.

Table 1
Residues for various initial values $\beta = \gamma = 0.0$, $a = -1.0$, $\delta = 0.1$

| $x(0)$ | $x'(0)$ | $R$ (R–K) | $\bar{R}$ (R–K) | $R$ | $\bar{R}$ |
|---|---|---|---|---|---|
| −0.60 | 0.00 | $0.506 \times 10^{-7}$ | $0.167 \times 10^{-7}$ | $0.510 \times 10^{-7}$ | $0.167 \times 10^{-7}$ |
| −0.20 | 0.00 | $0.169 \times 10^{-7}$ | $0.556 \times 10^{-8}$ | $0.170 \times 10^{-7}$ | $0.558 \times 10^{-8}$ |
| 0.20 | 0.00 | $0.169 \times 10^{-7}$ | $0.556 \times 10^{-8}$ | $0.170 \times 10^{-7}$ | $0.558 \times 10^{-8}$ |
| 0.60 | 0.00 | $0.506 \times 10^{-7}$ | $0.167 \times 10^{-7}$ | $0.510 \times 10^{-7}$ | $0.167 \times 10^{-7}$ |

Table 2
Computed values for one trajectory and $\beta = 1.0$, $\gamma = 0.5$, $a = -1.0$, $\delta = 0.1$, $\omega = 2.0$

| Time $t$ | R–K ($\Delta t = 10^{-3}$) | R–K ($\Delta t = 10^{-5}$) | Present method |
|---|---|---|---|
| 0.00 | −0.60000000000000 | −0.60000000000000 | −0.60000000000000 |
| 2.00 | 0.31409873313062 | 0.31371648688680 | 0.31371262566018 |
| 4.00 | 0.21999740058487 | 0.22044059534189 | 0.22044507101854 |
| 6.00 | −0.47055077080651 | −0.47067182673239 | −0.47067304947514 |
| 8.00 | 0.33360443088284 | 0.33340169440076 | 0.33339964678903 |
| 10.00 | 0.05954652826912 | 0.05991233804537 | 0.05991603265443 |
| 12.00 | −0.32062795324411 | −0.32080179335392 | −0.32080354965071 |
| 14.00 | 0.28719055817582 | 0.28711894859965 | 0.28711822548094 |
| 16.00 | −0.04153549375585 | −0.04128854903066 | −0.04128605432601 |
| 18.00 | −0.16776725136809 | −0.16794291861988 | −0.16794469393698 |
| 20.00 | 0.19691355029143 | 0.19694501148928 | 0.19694532965283 |
| 22.00 | −0.08725120643612 | −0.08713806567355 | −0.08713692209681 |
| 24.00 | −0.03334219372378 | −0.03347318448049 | −0.03347450902818 |
| 26.00 | 0.08095993897927 | 0.08105247526587 | 0.08105341076640 |
| 28.00 | −0.07810954785165 | −0.07812085613549 | −0.07812096968801 |

Table 3
Residues for various initial values $\beta = 1.0$, $\gamma = 0.5$, $a = -1.0$, $\delta = 0.1$, $\omega = 2.0$ and for $\Delta t = 10^{-3}$

| $x(0)$ | $x'(0)$ | $R$ (R–K) | $\bar{R}$ (R–K) | $R$ | $\bar{R}$ |
|---|---|---|---|---|---|
| −0.60 | 0.00 | $5.00 \times 10^{-4}$ | $3.17 \times 10^{-4}$ | $3.96 \times 10^{-7}$ | $1.47 \times 10^{-7}$ |
| −0.20 | 0.00 | $5.00 \times 10^{-4}$ | $3.17 \times 10^{-4}$ | $2.34 \times 10^{-7}$ | $1.42 \times 10^{-7}$ |
| 0.20 | 0.00 | $5.00 \times 10^{-4}$ | $3.17 \times 10^{-4}$ | $3.17 \times 10^{-7}$ | $1.45 \times 10^{-7}$ |
| 0.60 | 0.00 | $5.00 \times 10^{-4}$ | $3.17 \times 10^{-4}$ | $5.79 \times 10^{-7}$ | $1.58 \times 10^{-7}$ |

We know consider the case $\beta = 1.0$, $\gamma = 0.5$, $a = -1.0$, $\delta = 0.1$, $\omega = 2.0$, for which there is no analytical solution in closed form, as far as we could investigate. For this case, the R–K method is used for two time steps $\Delta t = 10^{-3}$ and $10^{-5}$. Our method is used for $T = 0.44$ and $N = 30$. The values for one trajectory are presented in Table 2. It is obtained that the computed values agree for 3 or 4 decimal digits for $\Delta t = 10^{-3}$ and for 5 or 6 for $\Delta t = 10^{-5}$. The $R$ and $\bar{R}$ estimated with $\Delta t = 10^{-3}$ for various initial values are presented in Table 3. It is obtained that the $R$ and $\bar{R}$ of the R–K method are 3 orders of magnitude greater than the corresponding ones of the presented method. It is also observed that the accuracy of the computed solution is reduced for all the initial values and is well above the $10^{-6}$ expected due to the discretization with central differences. Analogous results hold also for $\Delta t = 10^{-5}$. On the other hand, for the presented method the residues remain below the expected value of $10^{-6}$ and the accuracy of the solution is maintained.

The Duffing equation was studied for various values of the parameters and the initial values. In all cases, the results obtained by both methods are visually identical.

### *3.4. CPU time cost*

Concerning the required CPU time, we perform further calculations for the case $\alpha = -1$, $\beta = 1$, $\gamma = 0.5$, $\delta = 0.1$, $\omega = 2$. For the computation of the derivatives for the estimation of $R$, $\bar{R}$ we use the fourth order finite differences:

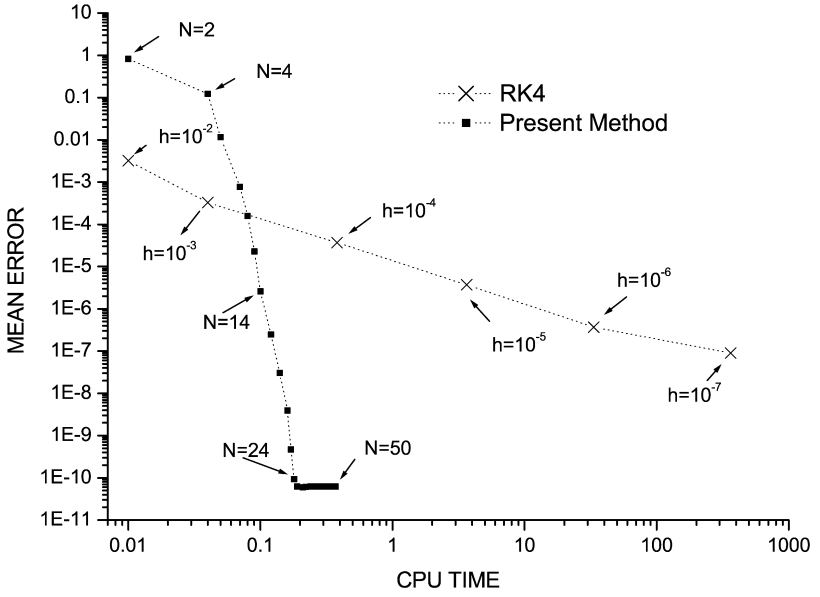


Fig. 1. Mean error, ($\bar{R}$), with respect to the required CPU time, for the forced non-linear dumped oscillator.

$$x'(t_i) = \frac{-x(t_{i+2}) + 8x(t_{i+1}) - 8x(t_{i-1}) + x(t_{i-2})}{12\Delta t} + O\big((\Delta t)^4\big),$$
$$x''(t_i) = \frac{-x(t_{i+2}) + 16x(t_{i+1}) - 30x(t_i) + 16x(t_{i-1}) - x(t_{i-2})}{12\Delta t^2} + O\big((\Delta t)^4\big).$$

The CPU time measurements where performed in a Pentium IV, 2.66 GHz, 512 DDR SDRAM using Compaq Visual FORTRAN 6.6.0 (Windows XP SP2) and the embedded subroutine "CPU TIME" before and after the block of calculations. We observed similar behavior for various initial points of integration and representative results are shown in Fig. 1 where $\bar{R}$ and the corresponding required CPU time computation are pictured, for initial values $x(0) = -0.6$, $x'(0) = 0.0$ and trajectory length of 100 time units. The $\bar{R}$ was estimated using the above relations and $\Delta t = 10^{-3}$ for the time interval [95, 100], for various time steps of integration for the R–K and for various $N$ for the present method. It is observed that for the R–K method the $\bar{R}$ reduces following $O(h)$ behavior. The robustness of the present method is clearly demonstrated in this figure where it is observed that the $\bar{R}$ reduces rapidly with the increase of $N$ and for $N = 24$ the minimum error (order of $10^{-9}$) detected by the finite differences method used, has been reached. The CPU time required by the present method is of the order of 0.2 s, for all $N$, whereas, for the R–K the CPU time is equivalent to 0.2 s for accuracy of something less than 4 decimal digits and increases to 365 s for 7 decimal digits. It worths to emphasize that accuracy of more than 7 decimal digits is practically inevitable by the R–K method whereas, is attained by the present method within 0.3 s. The difference in speed and accuracy between the R–K and the present method is growing up dramatically for the evaluation of very long trajectories. It is also obtained that accuracy of 7 decimal digits requires 3324.08 seconds CPU time whereas, accuracy of 11 decimal digits is attained within 1.5 s by the present method.

One explanation of the above behavior is the following: The R–K methods are obliged to evaluate the solution at each time step in order to proceed to the next step and this is the major practical difference of our method. The solution in the present method can be evaluated for steps

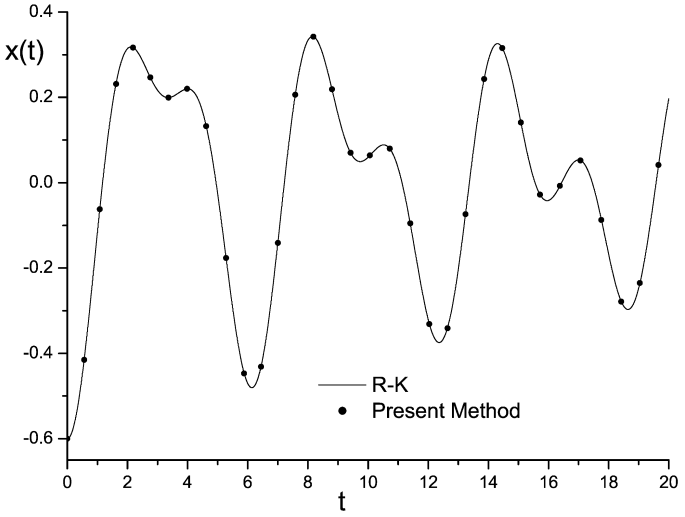


Fig. 2. $x(t)$ estimated by R–K and points of calculation of $x(t)$ by the present method, for the forced non-linear dumped oscillator.

equal to $T$. Moreover, due to the fact that the series of the coefficients tend to zero, the evaluation of the coefficients from (2.7) can be interrupted if some sequential coefficients are bellow the round of error, i.e. $10^{-14}$. After numerous numerical experiments it has been observed that $T$ varies along the trajectory from 0.4 to 0.7 and $N$ from 26 to 34. Experiments have also been performed for $T = 0.1$ and $N = 100$ and no differences up to 13 decimal digits have been found. Thus, using our method, the trajectory is estimated with round of error accuracy with "jumps," e.g. of 0.5 on the trajectory, whereas with the R–K method the solution is estimated at each time step e.g. $10^{-5}$. This is shown in Fig. 2.

## 4. Application to the Lorenz system

### *4.1. Derivation of the discrete equivalent system*

In this section we shall apply the presented method to the Lorenz system:

$$\left.\begin{aligned} x'(t) &= -\sigma\big[x(t) - y(t)\big] \\ y'(t) &= -x(t)s(t) + rx(t) - y(t) \\ s'(t) &= x(t)y(t) - \beta s(t) \end{aligned}\right\} \quad |t| < T,\ T > 0,\ \sigma, r, \beta \in \mathbb{R}, \tag{4.1}$$

$$x(0) = x^0, \qquad y(0) = y^0, \qquad s(0) = s^0. \tag{4.2}$$

The system (4.1) has been proposed by Lorenz in [29], in his attempt to find a model describing weather forecasting. Actually, system (4.1) is a model of two-dimensional convection in a horizontal layer of fluid heated from below and cooled from above, where $x$ stands for the velocity of the fluid and $y$, $s$ stand for its temperature. For more details see [22, pp. 233–246], [30, pp. 194–199].

First, using $z = \frac{t}{T}$ we restrict the problem to $\mathbb{D}$ and (4.1)–(4.2) becomes:

$$\left.\begin{aligned} x'(z) &= -\sigma T\big[x(z)-y(z)\big] \\ y'(z) &= T\big[-x(z)s(z)+rx(z)-y(z)\big] \\ s'(z) &= T\big[x(z)y(z)-\beta s(z)\big] \end{aligned}\right\} \quad |z|<1, \tag{4.3}$$

$$x(z=0)=x^0, \qquad y(z=0)=y^0, \qquad s(z=0)=s^0. \tag{4.4}$$

As in Section 3.1, using Propositions 2.1 and 2.3, the abstract form of (4.3) in $H_1^3 = H_1 \times H_1 \times H_1$ can be found to be:

$$\left.\begin{aligned} C_0V^*x &= -\sigma T(x-y) \\ C_0V^*y &= T\big(-x_1(V)s+rx-y\big) \\ C_0V^*s &= T\big(x_1(V)y-\beta s\big) \end{aligned}\right\}. \tag{4.5}$$

If we take the inner product of both parts of (4.5), with $e_n$ we obtain:

$$\left.\begin{aligned} \langle C_0V^*x, e_n\rangle &= -\sigma T\big[\langle x,e_n\rangle - \langle y,e_n\rangle\big] \\ \langle C_0V^*y, e_n\rangle &= T\big[-\langle x_1(V)s,e_n\rangle + r\langle x,e_n\rangle - \langle y,e_n\rangle\big] \\ \langle C_0V^*s, e_n\rangle &= T\big[\langle x_1(V)y,e_n\rangle - \beta\langle s,e_n\rangle\big] \end{aligned}\right\}$$

or eventually

$$\left.\begin{aligned} x_{n+1} &= -\frac{\sigma T}{n}(x_n-y_n) \\ y_{n+1} &= \frac{T}{n}\left[-\sum_{k=1}^{n} x_k s_{n-k+1} + rx_n - y_n\right] \\ s_{n+1} &= \frac{T}{n}\left[\sum_{k=1}^{n} x_k y_{n-k+1} - \beta s_n\right] \end{aligned}\right\} \quad n=1,2,\ldots, \tag{4.6}$$

which is the discrete equivalent system to (4.1). Using (4.4), we find as in Section 3.2 that the initial conditions accompanying (4.6) are:

$$x_1 = x^0, \qquad y_1 = y^0, \qquad s_1 = s^0. \tag{4.7}$$

Using (4.6)–(4.7) we find the coefficients $x_n$, $y_n$, $s_n$ of the solution of (4.1)–(4.2)

$$x(t) = \sum_{n=1}^{\infty} x_n \left(\frac{t}{T}\right)^{n-1}, \qquad y(t) = \sum_{n=1}^{\infty} y_n \left(\frac{t}{T}\right)^{n-1},$$

$$s(t) = \sum_{n=1}^{\infty} s_n \left(\frac{t}{T}\right)^{n-1}, \quad |t|<T.$$

### *4.2. Existence and uniqueness conditions*

Now we will study the system (4.5) following the main ideas introduced in [3], [21] and using the Banach space $H_1^3$ with vector elements $\tilde{f} = (x,y,s)$, where $x,y,s \in H_1$ and norm $\|\tilde{f}\|_1^2 = \|x\|_1^2 + \|y\|_1^2 + \|s\|_1^2$. Let $D = \operatorname{diag}(d_1,d_2,d_3)$ where $d_i$, $i=1,2,3$, non-negative integers. Define the operators $\Omega_D$, $\Omega_D^*$ on $H_1^3$ as: $\Omega_D\tilde{f} = (V^{d_1-1}x, V^{d_2-1}y, V^{d_3-1}s)$, $\Omega_D^*\tilde{f} = (V^{*^{d_1-1}}x, V^{*^{d_2-1}}y, V^{*^{d_3-1}}s)$, where $V^{d_i-1} = I = (V^*)^{d_i-1}$, for $d_i = 1$, $V^{d_i-1} = V^*$ for $d_i = 0$, and $(V^*)^{d_i-1} = V$ for $d_i = 0$. The norms of these operators in $H_1^3$ are $\|\Omega_D\|_1 = \|\Omega_D^*\|_1 = 1$. For the system (4.5) the following holds:

**Theorem 4.1.** *Let*

$$T\max\{2|\sigma|,|r|+1,|\beta|\}<1, \tag{4.8}$$

$$|x^0|^2+|y^0|^2+|s^0|^2<\frac{(1-T\max\{2|\sigma|,|r|+1,|\beta|\})^4}{16T^2}, \tag{4.9}$$

*where $\sigma$, $\beta$, $r$, $x^0$, $y^0$ and $s^0$, are the parameters and the initial conditions of* (4.1)*. Then the system* (4.5) *has a unique solution in $H_1^3$ which is bounded by $\frac{1-T\max\{2|\sigma|,|r|+1,|\beta|\}}{2T}$. Equivalently, the system* (4.6)–(4.7) *has a unique solution in $\ell_1^3$ and the system* (4.3)–(4.4) *has a unique solution in $[H_1(\mathbb{D})]^3$. Moreover, the following holds*:

$$\left(\sum_{n=1}^{\infty}|x_n|\right)^2+\left(\sum_{n=1}^{\infty}|y_n|\right)^2+\left(\sum_{n=1}^{\infty}|s_n|\right)^2<\left(\frac{1-T\max\{2|\sigma|,|r|+1,|\beta|\}}{2T}\right)^2.$$

**Remark 4.2.** The preceding theorem holds also when $z\in\mathbb{C}$. The only difference would be that the appearing absolute values would become moduli.

**Proof.** The system of operator equations (4.5) can also be written as

$$\tilde{C}_0\Omega_D^*\tilde{f}=TA\tilde{f}+T\tilde{N}(\tilde{f}), \tag{4.10}$$

where

$$\tilde{f}=(x,y,s),\qquad \tilde{N}(\tilde{f})=\big(0,-x_1(V)s,x_1(V)y\big),\qquad \tilde{C}_0(\tilde{f})=(C_0x,C_0y,C_0s),$$

$$\Omega_D^*\tilde{f}=(V^*x,V^*y,V^*s),\qquad A=\begin{pmatrix}-\sigma & \sigma & 0\\ r & -1 & 0\\ 0 & 0 & -\beta\end{pmatrix}.$$

Then (4.10) takes the form

$$\Omega_D^*\tilde{f}=TA\tilde{B}_0\tilde{f}+T\tilde{B}_0\tilde{N}(\tilde{f})\quad\Rightarrow\quad \tilde{f}=TA\Omega_D\tilde{B}_0\tilde{f}+T\Omega_D\tilde{B}_0\tilde{N}(\tilde{f})+\tilde{h}, \tag{4.11}$$

where $\tilde{B}_0(\tilde{f})=(B_0x,B_0y,B_0s)$, $\tilde{h}=(c_1e_1,c_2e_1,c_3e_1)$ and $c_1$, $c_2$, $c_3$ are arbitrary constants which are found by taking the inner product of both parts of (4.11) with $e_1$. Indeed $c_1=(x,e_1)=x_1=x^0$. Similarly $c_2=y^0$, $c_3=s^0$ and (4.11) becomes:

$$(I-TA\Omega_D\tilde{B}_0)\tilde{f}=T\Omega_D\tilde{B}_0\tilde{N}(\tilde{f})+\tilde{h}. \tag{4.12}$$

According to Theorem 3.3 and condition (4.2), the following holds:

$$\big\|(I-TA\Omega_D\tilde{B}_0)^{-1}\big\|_1<\frac{1}{1-T\max\{2|\sigma|,|r|+1,|\beta|\}}.$$

(We have considered the norm of $A=(a_{ij})$ to be $\|A\|=\max_i\sum_j|a_{ij}|$.) Then the system (4.12) becomes:

$$\tilde{f}=(I-TA\Omega_D\tilde{B}_0)^{-1}\big[\tilde{h}+T\Omega_D\tilde{B}_0\tilde{N}(\tilde{f})\big]=\phi(\tilde{f}). \tag{4.13}$$

Let $\|\tilde{f}\|_1\leqslant R$, $R$ sufficiently large but finite. Then $\|x\|_1+\|y\|_1+\|s\|_1\leqslant R\Rightarrow\|x\|_1,\|y\|_1+\|s\|_1\leqslant R$. Thus

$$\big\|\tilde{N}(\tilde{f})\big\|_1\leqslant\|x\|_1\cdot\|s\|_1+\|x\|_1\cdot\|y\|_1=\|x\|_1\cdot\big(\|s\|_1+\|y\|_1\big)\leqslant R^2.$$

As a consequence we obtain from (4.13):

$$\left\|\phi(\tilde{f})\right\|_1 \leqslant \frac{\|\tilde{h}\|_1}{1 - T\max\{2|\sigma|, |r|+1, |\beta|\}} + \frac{T}{1 - T\max\{2|\sigma|, |r|+1, |\beta|\}} R^2.$$

Let

$$P(R) = R - \frac{T}{1 - T\max\{2|\sigma|, |r|+1, |\beta|\}} R^2.$$

This function has the maximum

$$P(R_0) = \frac{R_0}{2} \quad \text{at } R_0 = \frac{1 - T\max\{2|\sigma|, |r|+1, |\beta|\}}{2T}.$$

Then, if for arbitrary $\epsilon > 0$ it is

$$\frac{\|\tilde{h}\|_1}{1 - T\max\{2|\sigma|, |r|+1, |\beta|\}} \leqslant P(R_0) - \epsilon,$$

we have $|\phi(\tilde{f})| \leqslant R_0 - \epsilon < R_0$. Moreover, the mapping $\phi(\tilde{f})$ is Fréchet differentiable in $S(0, R_0) = \{\tilde{f} \in H_1^3\colon \|\tilde{f}\|_1 < R_0\}$, since the non-linear operator $\tilde{N}(\tilde{f})$ is Fréchet differentiable. Thus Theorem 3.4 can be applied to (4.13) and this completes the proof of the theorem. □

### 4.3. Numerical results

For the numerical results we chose the values $r = 28$, $\sigma = 10$, $\beta = 8/3$, $x^0 = 0$, $y^0 = 1$, $s^0 = 0$, which were also used by Lorenz in [29]. Following the analysis presented in Section 3.3 we solve with the present method successive initial value problems (for $T = 0.002$, $N = 10$)

Table 4
Computed values for one trajectory and $\sigma = 10.0$, $r = 28.0$, $\beta = 8.0/3.0$

| Method | Time $t$ | $x$ | $y$ | $s$ |
|---|---|---|---|---|
| | 0.0 | 0.00000000000000 | 1.00000000000000 | 0.00000000000000 |
| Present | 4.0 | −9.22656909651577 | −10.37681167457006 | 26.39546104450844 |
| R–K ($\Delta t = 10^{-3}$) | | −9.22656909092673 | −10.37681166773029 | 26.39546103871058 |
| R–K ($\Delta t = 10^{-5}$) | | −9.22656909651611 | −10.37681167456993 | 26.39546104450971 |
| Present | 8.0 | −7.80426232466933 | −5.84857978422391 | 28.64545320069553 |
| R–K ($\Delta t = 10^{-3}$) | | −7.80426233663573 | −5.84857979833848 | 28.64545321081120 |
| R–K ($\Delta t = 10^{-5}$) | | −7.80426232466823 | −5.84857978422352 | 28.64545320069344 |
| Present | 12.0 | −5.61586252258209 | −6.93058832663161 | 21.21243268024443 |
| R–K ($\Delta t = 10^{-3}$) | | −5.61586253600767 | −6.93058833739967 | 21.21243270477625 |
| R–K ($\Delta t = 10^{-5}$) | | −5.61586252258421 | −6.93058832663542 | 21.21243268024400 |
| Present | 16.0 | −1.33958050847073 | −1.33905207761989 | 18.08308447491430 |
| R–K ($\Delta t = 10^{-3}$) | | −1.33958059384526 | −1.33905224922778 | 18.08308430318110 |
| R–K ($\Delta t = 10^{-5}$) | | −1.33958050847619 | −1.33905207763238 | 18.08308447489753 |
| Present | 24.0 | 11.94527374082624 | 17.23486786028362 | 24.47713891383506 |
| R–K ($\Delta t = 10^{-3}$) | | 11.94513246479799 | 17.23500674394074 | 24.47643707855276 |
| R–K ($\Delta t = 10^{-5}$) | | 11.94527373157132 | 17.23486786939897 | 24.47713886783172 |
| Present | 28.0 | −9.23868922295957 | −2.20585494120638 | 34.94761658104625 |
| R–K ($\Delta t = 10^{-3}$) | | −9.23155611824196 | −2.20736741527057 | 34.93361538414999 |
| R–K ($\Delta t = 10^{-5}$) | | −9.23868875182012 | −2.20585503706215 | 34.94761565957072 |

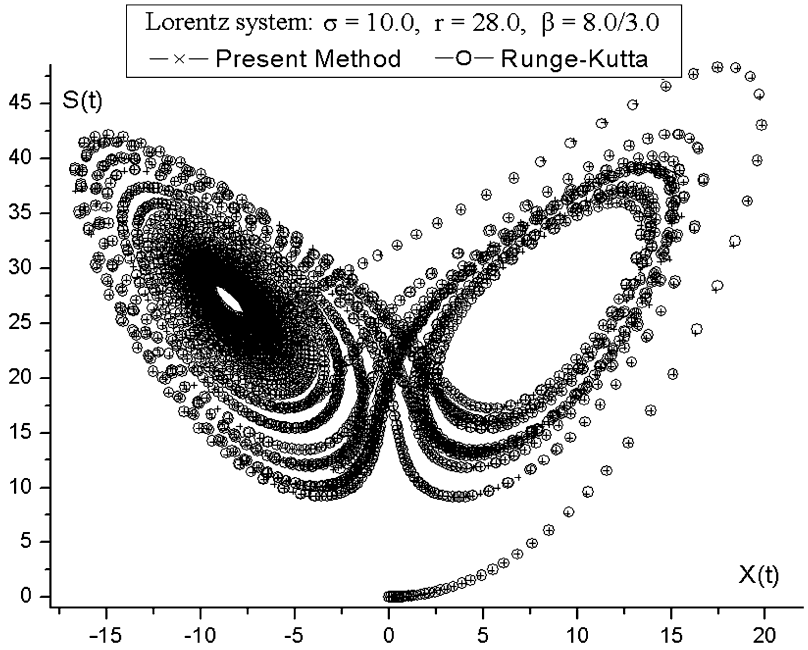


Fig. 3. Graph of the computed trajectory in the phase plane $x(t)$–$s(t)$.

in order to obtain trajectories in the phase plane of length 30 time units. The values for one trajectory estimated using both R–K and the proposed method are presented in Table 4. The R–K method is used for two time steps $\Delta t = 10^{-3}$ and $10^{-5}$. For the R–K method with $\Delta t = 10^{-3}$ the computed values agree for 7 or 8 decimal digits with those obtained with the present method for short times. As time elapses the values agree for fewer decimal digits. When the time step of the R–K is further decreased to $\Delta t = 10^{-8}$ the computed values agree with the present method from 11 to 12 decimal digits for short times. The residues obtained for both methods where of about the same order and no clear conclusion could be obtained from this. This is possibly caused due to the central differences used for the estimation of the derivatives of the estimated variables, which may introduce an error large enough to diminish the difference between the two methods. Nevertheless, as it is also shown in the indicative Table 4 the solution estimated with the R–K method "converges" to the solution obtained by our method as the time step decreases. Indicatively, the computed trajectories in the phase planes $x - s$ for $x(0) = 0$, $y(0) = 1$, $s(0) = 0$ are shown in Fig. 3.

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